Complex Oscillators

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1 Complex Oscillators

Intuitively, if we think about what quantum field theory is trying to do, it makes sense that the field should consist of complex harmonic oscillators, that is oscillators that oscillate in some non-spatial complex coordinate. [1] This is because the field is supposed to contain information about particle wave functions, which are complex-valued functions.
Therefore, we should be trying to solve the complex simple harmonic oscillator Schrodinger equation.

$$-\frac{1}{2}\frac{\partial}{\partial s}\frac{\partial}{\partial s^*}\psi_n(s)+\frac{1}{2}\omega^2|s|^2\psi_n(s)=E_n\psi_n$$

where we treat s and s^* as independent parameters. Now based on our previous observations, we expect that the excited states will be powers of s^* times the ground state Gaussian, so our Ansatz will be

$$\psi_n(s)=(s^*)^n e^{-\frac{1}{2}\omega|s|^2}$$

We perform the derivatives

$$\frac{\partial}{\partial s}\frac{\partial}{\partial s^*}\psi_n(s)=\frac{\partial}{\partial s}\left(n(s^*)^{n-1}e^{-\frac{1}{2}\omega|s|^2}+(s^*)^n\left(-\frac{1}{2}\omega s\right)e^{-\frac{1}{2}\omega|s|^2}\right)$$

$$=n(s^*)^{n-1}\left(-\frac{1}{2}\omega s^*\right)e^{-\frac{1}{2}\omega|s|^2}+(s^*)^n\left(-\frac{1}{2}\omega\right)e^{-\frac{1}{2}\omega|s|^2}+(s^*)^n\left(-\frac{1}{2}\omega s\right)\left(-\frac{1}{2}\omega s^*\right)e^{-\frac{1}{2}\omega|s|^2}$$

$$=\left(\frac{1}{4}\omega^2|s|^2-\frac{n+1}{2}\right)\psi_n(s)$$

Rearranging we get

$$-\frac{1}{2}\frac{\partial}{\partial s}\frac{\partial}{\partial s^*}\psi_n(s)+\frac{1}{8}\omega^2|s|^2\psi_n(s)=\frac{n+1}{2}\psi_n(s)$$

We can see that our Ansatz is a solution to the complex simple harmonic oscillator Schrodinger equation if we wrap the factor of $\frac{1}{2}$ into ω.

2 Ehrenfest Theorem

In a previous lecture, we showed that the Ehrenfest theorem implies

$$\left\langle\Psi\right|\left(\hat{O}\hat{\phi}\right)\left|\Psi\right\rangle=0$$

where $\hat{O}$ is the Klein-Gordon equation operator,

$$\hat{O}=\frac{\partial^2}{\partial t^2}-\nabla^2+m^2$$

So if we multiply both sides of the result by $|\phi\rangle\langle\phi|$ we obtain

$$\langle\Psi|\ \phi\rangle\left(\hat{O}\langle\phi|\ \hat{\phi}\right)|\Psi\rangle=0$$

[1] Perhaps we should use w for the non-spatial coordinate since this letter is more commonly associated with complex numbers.

where we were able to move $\langle\phi|$ past $\hat{O}$ because their commutator is zero because $\langle\phi|$ does not depend on space or time coordinates.

$$\langle\Psi|\ \phi\rangle \left(\hat{O}\phi\right) \langle\phi|\ \Psi\rangle = 0$$

$$\left(\hat{O}\phi\right) |\Psi|^2[\phi] = 0$$

Therefore we see that either ϕ satisfies the Klein-Gordon equation or $\Psi[\phi] = 0$. This means that the Schrodinger functional only accepts fields that obey the field equation. Any field that does not obey the field equation always has a zero probability of manifesting.

3 Wave Function Evolution

We can show that the previous result implies that particle wave functions evolve according to the field equation.
The wave function is defined implicitly by

$$\Psi_1[\phi] = \int d^3x\ \psi(\mathbf{x},t)\phi^*(\mathbf{x},t)\Psi_0[\phi]$$

This is a solution to the functional Schrodinger equation, so

$$\hat{H} \int d^3x\ \psi(\mathbf{x},t)\phi^*(\mathbf{x},t)\Psi_0[\phi] = i\frac{\partial}{\partial t} \int d^3x\ \psi(\mathbf{x},t)\phi^*(\mathbf{x},t)\Psi_0[\phi]$$

Since $\hat{H}$ contains two derivatives with respect to fields, it commutes with ϕ^*. [2]

$$\int d^3x\ \psi(\mathbf{x},t)\phi^*(\mathbf{x},t) \left(\hat{H}\Psi_0[\phi]\right) = i\int d^3x\ \frac{\partial}{\partial t} \left(\psi(\mathbf{x},t)\phi^*(\mathbf{x},t)\right) \Psi_0[\phi] + \int d^3x\ \psi(\mathbf{x},t)\phi^*(\mathbf{x},t) \left(i\frac{\partial}{\partial t}\Psi_0[\phi]\right)$$

Since $\Psi_0[\phi]$ is also a solution to the functional Schrodinger equation, we can cancel the first and last terms on the previous line.

$$0 = i\int d^3x\ \frac{\partial}{\partial t} \left(\psi(\mathbf{x},t)\phi^*(\mathbf{x},t)\right) \Psi_0[\phi]$$

Now since the value of the wave function at different points in space can be modified independently[3], we must have that the integrand is zero everywhere.

$$0 = i\frac{\partial}{\partial t} \left(\psi(\mathbf{x},t)\phi^*(\mathbf{x},t)\right)$$

$$0 = i\psi(\mathbf{x},t)\frac{\partial}{\partial t}\phi^*(\mathbf{x},t) + i\phi^*(\mathbf{x},t)\frac{\partial}{\partial t}\psi(\mathbf{x},t)$$

Now if we treat the Klein-Gordon equation like a Schrodinger equation, meaning that we pretend that we have an expression for $\hat{H}$ even though it is a mess containing square roots of derivative operators, then

$$\hat{H}_{KG}\phi = i\frac{\partial}{\partial t}\phi$$

Taking the Hermitian conjugate of both sides,

[2]Problem: This is not rigorous. You could use it to say that $\hat{H}$ commutes with ϕ^2 by saying that it commutes with each of the two factors.

[3]It will also change the time dependence so as to satisfy the field equation, but we only care about spatial dependence because that is what we are integrating over.

$$\phi^* \hat{H}_{KG}^\dagger = -i\frac{\partial}{\partial t}\phi^*$$

Plugging this into the previous result,

$$\left(\phi^*(\mathbf{x},t)\hat{H}_{KG}^\dagger\right)\psi(\mathbf{x},t) = i\phi^*(\mathbf{x},t)\frac{\partial}{\partial t}\psi(\mathbf{x},t)$$

$$\phi^*(\mathbf{x},t)\left(\hat{H}_{KG}\psi(\mathbf{x},t)\right) = i\phi^*(\mathbf{x},t)\frac{\partial}{\partial t}\psi(\mathbf{x},t)$$

$$\hat{H}_{KG}\psi(\mathbf{x},t) = i\frac{\partial}{\partial t}\psi(\mathbf{x},t)$$

So $\psi(\mathbf{x},t)$ evolves according to the same equation as $\phi(\mathbf{x},t)$, namely the Klein-Gordon equation.

4 Note

We had a debate over whether to call the point oscillator wave functions "functionals" because they depend on the field ϕ. I believe that is is misleading to call them functionals because the dependence on ϕ only appears due to the fact that the Hamiltonian in their Schrodinger equation depends on ϕ.

$$\hat{H}^\phi\psi_{\mathbf{x}}(s) = i\frac{\partial}{\partial t}\psi_{\mathbf{x}}(s)$$

To call $\psi_{\mathbf{x}}(s)$ a functional confuses the fact that it is really just a plain old quantum mechanical wave function with a complex parameter.