Ground State

Chris Clark April 12, 2008

In this paper we will show that the following equation

$$\Phi[\phi] = e^{\int \ln(\psi^{\phi}_{\mathbf{x}}(\phi(\mathbf{x})))\ d^3x}$$

yields the same ground state Schrodinger functional as is found in Hatfield equation (10.26) if we assume that the one-point harmonic oscillator wave functions are Gaussians, as is the case for an independent oscillator.

First of all, we can determine the dimension of ϕ by dimensional analysis on the Hamiltonian. If we assume a Klein-Gordon Hamiltonian,

$$\hat{H} = \frac{1}{2}\int d^3x' \left(-\frac{\delta^2}{\delta\phi^2(\mathbf{x}')} + |\boldsymbol{\nabla}'\phi(\mathbf{x}')|^2 + m^2\phi^2(\mathbf{x}') \right)$$

we find by comparing the dimensions of the first and last term that $[1/\phi^2] = [m^2\phi^2]$ so $[\phi^2] = [1/m]$. However, there is no mass directly involved with the one-point oscillators. The mass is just a parameter in the field equation (the Klein Gordon equation in this case). So the only way to cancel the units is if the variance of the Gaussian takes on a dimension of inverse energy. This is entirely analogous to the case of simple harmonic oscillators in quantum mechanics.

The Gaussian will have the form

$$\psi^{\phi}_{\mathbf{x}}(s) = e^{-\frac{1}{2}\omega^{\phi}(\mathbf{x})s^2}$$

where $\omega^{\phi}(\mathbf{x})$ is the energy at the point $\mathbf{x}$ given the field configuration ϕ. The energy is based on the frequency of the field oscillation mode. Each point in the field will have some superposition of modes based on the Fourier decomposition of the field at that point. Since we are assuming the field obeys the Klein-Gordon equation, we will have $\omega(\mathbf{q}) = \sqrt{|\mathbf{q}|^2 + m^2}$. Therefore the energy at a specific point is given by the weighted sum [1]

$$\omega^{\phi}(\mathbf{x}) = \frac{\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}}\omega(\mathbf{q})\tilde{\phi}(\mathbf{q})}{\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}}\tilde{\phi}(\mathbf{q})}$$

Therefore

$$\ln(\psi^{\phi}_{\mathbf{x}}(\phi(\mathbf{x}))) = -\frac{1}{2}\omega^{\phi}(\mathbf{x})\phi^2(\mathbf{x}) = -\frac{1}{2}\left[\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}}\omega(\mathbf{q})\tilde{\phi}(\mathbf{q})\right]\phi(\mathbf{x})$$

So by inserting the Fourier transform for $\phi(\mathbf{x})$ and plugging into the definition of $\Phi[\phi]$, [2]

$$\begin{aligned}
\Phi[\phi] &= \exp\left[-\frac{1}{2}\int d^3x \ \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}}\omega(\mathbf{q})\tilde{\phi}(\mathbf{q}) \int \frac{d^3q'}{(2\pi)^3} e^{i\mathbf{q}'\cdot\mathbf{x}}\tilde{\phi}(\mathbf{q}')\right] \\
&= \exp\left[-\frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\int \frac{d^3q'}{(2\pi)^3}\omega(\mathbf{q})\tilde{\phi}(\mathbf{q})\tilde{\phi}(\mathbf{q}') \int d^3x\ e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}}\right] \\
&= \exp\left[-\frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\int \frac{d^3q'}{(2\pi)^3}\omega(\mathbf{q})\tilde{\phi}(\mathbf{q})\tilde{\phi}(\mathbf{q}')(2\pi)^3\delta(\mathbf{q}+\mathbf{q}')\right] \\
&= \exp\left[-\frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\omega(\mathbf{q})\tilde{\phi}(\mathbf{q})\tilde{\phi}(-\mathbf{q})\right]
\end{aligned}$$

[1] This part needs a more complete explanation, it was discovered by reverse-engineering the result in Hatfield.

[2] This part was inspired by the paper “Precanonical quantization and the Schrodinger wave functional” by Kanatchikov.