Ground State Part 2

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Our goal is to find the ground state wave function for the one-point oscillators. We will accomplish this by rewriting the Hamiltonian for the Schrodinger functional as an integral over simple harmonic oscillator Hamiltonians. We start with the Klein-Gordon Hamiltonian

$$\hat{H} = \frac{1}{2}\int d^3x \left(-\frac{\delta^2}{\delta\phi^2(\mathbf{x})} + (\boldsymbol{\nabla}\phi(\mathbf{x}))^2 + m^2\phi^2(\mathbf{x}) \right)$$

So the Hamiltonian density, corresponding to the Hamiltonian for a one-point oscillator, is

$$\hat{\mathcal{H}}(\mathbf{x}) = \frac{1}{2}\left(-\frac{\delta^2}{\delta\phi^2(\mathbf{x})} + (\boldsymbol{\nabla}\phi(\mathbf{x}))^2 + m^2\phi^2(\mathbf{x}) \right)$$

We now define

$$\hat{\mathcal{H}}'(\mathbf{x}) \equiv \frac{1}{2}(\boldsymbol{\nabla}\phi(\mathbf{x}))^2 + \frac{1}{2}m^2\phi^2(\mathbf{x})$$

which is the non-momentum part of the Hamiltonian density. Now we replace ϕ with its Fourier transform in order to simplify the gradient.

$$\phi(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}}\tilde{\phi}(\mathbf{q})$$

Therefore,

$$\boldsymbol{\nabla}\phi(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}}(i\mathbf{q})\tilde{\phi}(\mathbf{q})$$

$$(\boldsymbol{\nabla}\phi(\mathbf{x}))^2 = \int \frac{d^3q}{(2\pi)^3}\frac{d^3q'}{(2\pi)^3} e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}}(-\mathbf{q}\cdot\mathbf{q}')\tilde{\phi}(\mathbf{q})\tilde{\phi}(\mathbf{q}')$$

$$m^2\phi^2(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3}\frac{d^3q'}{(2\pi)^3} e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}} m^2\tilde{\phi}(\mathbf{q})\tilde{\phi}(\mathbf{q}')$$

$$\hat{\mathcal{H}}'(\mathbf{x}) = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\frac{d^3q'}{(2\pi)^3} e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}}(m^2 - \mathbf{q}\cdot\mathbf{q}')\tilde{\phi}(\mathbf{q})\tilde{\phi}(\mathbf{q}')$$

Now we need to pull a little trick. We recall that this Hamiltonian density is integrated over all space. Therefore, the exponential will become $(2\pi)^3\delta^3(\mathbf{q}+\mathbf{q}')$. So we can make the replacement $\mathbf{q}' = -\mathbf{q}$ without affecting the resulting physics. It is important to note however that this is really an effectively equivalent Hamiltonian, but not actually equivalent. It is only provides equivalent results after integrating over all space, so one must be careful not to use this effective Hamiltonian for a single point oscillator.

$$\hat{\mathcal{H}}'(\mathbf{x}) = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\frac{d^3q'}{(2\pi)^3} e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}}(m^2 + \mathbf{q}\cdot\mathbf{q})\tilde{\phi}(\mathbf{q})\tilde{\phi}(\mathbf{q}')$$

$$\hat{\mathcal{H}}'(\mathbf{x}) = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\frac{d^3q'}{(2\pi)^3} e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}}\sqrt{m^2+q^2}\,\tilde{\phi}(\mathbf{q})\sqrt{m^2+q'^2}\,\tilde{\phi}(\mathbf{q}')$$

$$\hat{\mathcal{H}}'(\mathbf{x}) = \frac{1}{2}\left[\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}}\sqrt{m^2+q^2}\,\tilde{\phi}(\mathbf{q})\right]^2$$

$$\hat{\mathcal{H}}'(\mathbf{x}) = \frac{1}{2}\left[\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}}\omega(\mathbf{q})\tilde{\phi}(\mathbf{q})\right]^2$$

where we have defined $\omega(\mathbf{q}) \equiv \sqrt{m^2+q^2}$. Finally we multiply and divide by ϕ^2 to obtain

$$\hat{\mathcal{H}}'(\mathbf{x}) = \frac{1}{2}\left[\frac{\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}}\omega(\mathbf{q})\tilde{\phi}(\mathbf{q})}{\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}}\tilde{\phi}(\mathbf{q})}\right]^2 \phi^2(\mathbf{x})$$

We know define the quantity inside the square brackets to be $\omega_\phi(\mathbf{x})$. Therefore we can write

$$\hat{\mathcal{H}}(\mathbf{x}) = \frac{1}{2}\hat{\pi}^2(\mathbf{x}) + \frac{1}{2}\omega_\phi^2(\mathbf{x})\phi^2(\mathbf{x})$$

which we recognize as a simple harmonic oscillator Hamiltonian at each point in space. The ground state of a simple harmonic oscillator is given in Sakurai (2.3.30)

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

In this case, m is wrapped up into ϕ, so we can simply drop m from this formula to obtain the result for the one-point oscillator ground state wave function.

$$\psi_{\mathbf{x}}^{\phi}(s) = \left(\frac{\omega_\phi(\mathbf{x})}{\pi\hbar}\right)^{1/4} e^{-\frac{1}{2\hbar}\omega_\phi(\mathbf{x})s^2}$$