Uniqueness of the QED Lagrangian

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1 Introduction

It is said that the QED Lagrangian desnity[1] is uniquely determined by the conditions

(a) Locality - $\mathcal{L}(x)$ only depends on the fields at x, so we can't do $\mathcal{L}(x) = m\bar{\psi}(x)\psi(x+c)$

(b) Lorentz Invariance

(c) P or T Invariance

(d) Local Gauge Invariance

(e) Renormalizability

What does renormalizability mean? Peskin and Schroeder state on page 481: "...the Lagrangian of a renormalizable theory can contain no terms of mass dimension higher than 4". This is deceptive because all terms must have a mass dimension of four in order for the Lagrangian density to have the units of an energy density. They implicitly intend the reader to count the mass dimension by neglecting the dimensions of the coupling constants. This statement can be rephrased as found in Aitchison and Hey page 154, "...theories in which the coupling constant has negative mass dimensions, such as the 'four fermion' theory, are not renormalizable."

2 Local Gauge Invariance

See Peskin and Schroeder section 15.1 where they derive

$$D_\mu = \partial_\mu + ieA_\mu(x)$$

is the covariant derivative on $\psi(x)$, where $A_\mu(x)$ is called the connection, or the gauge field, which has the transformation property

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$$

under the local gauge transformation

$$\psi(x) \to e^{i\alpha(x)}\psi(x)$$

Note: The quanta of the gauge field are called gauge bosons.

3 Listing Possible Terms

Each term must satisfy all the constraints listed in the introduction, in addition to

(a) Being scalar

(b) Having mass dimension 4 so that the action $S = \int \mathcal{L}\ d^4x$ is dimensionless.

[1] We will sometimes use 'Lagrangian' in place of 'Lagrangian density' as is commonly found in the literature.

Note that $[\psi] = 3/2, [\phi] = 1, [A^\mu] = 1, [\partial_\mu] = 1, [x] = -1, [m] = 1$ (see Peskin and Schroeder page 80, section 4.1). The ordering of cuts is helpful. First we will generate a list of all possible terms that satisfy locality, scalar, renormalizability, and gauge invariance. Then we can apply mass dimension and Lorentz invariance to determine constants. Finally we apply P or T invariance to make the final cut.

Rule: The connection cannot show up unless if it is inside of a covariant derivative. We should try to prove that that is the *only* way it can be gauge invariant. The concept is that if you want to have local gauge invariance, then you can't introduce derivatives alone or the connection alone, they must come together in the proper form of the covariant derivative.

We can split the possible terms into two categories, those that contain ψ and those that don't contain ψ. At first we don't specify constants. We can fill them in later based on dimension counting and Lorentz invariance. Constants will be written as S, V, M for scalar, vector, and matrix.

Case 1: Terms that do not contain ψ

If a term does not contain $\psi(x)$, then it must contain the connection $A_\mu(x)$ or else it would be a trivial constant since otherwise it would be independent of the coordinate x. But as we argued earlier, the connection must come in the form of the covariant derivative. So we must construct a scalar out of the covariant derivative. But we can't have a free derivative in the term because there is no wave function to act it on. We can get around this problem by using the fact that derivatives commute. We calculate the commutator

$$[D_\mu, D_\nu] = [\partial_\mu + ieA_\mu(x), \partial_\nu + ieA_\nu(x)] = [\partial_\mu, \partial_\nu] + ie[\partial_\mu, A_\nu(x)] + ie[A_\mu(x), \partial_\nu] - e^2[A_\mu(x), A_\nu(x)]$$

Partial derivatives commute, and scalar fields such as A commute so

$$[D_\mu, D_\nu]\psi(x) = ie\left\{\partial_\mu(A_\nu(x)\psi(x)) - A_\nu(x)\partial_\mu\psi(x) + A_\mu(x)\partial_\nu\psi(x) - \partial_\nu(A_\mu(x)\psi(x))\right\}$$

$$[D_\mu, D_\nu] = ie(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))$$

But now the derivatives are bound, so this is not an operator, it is just a function.

$$[D_\mu, D_\nu] = ieF_{\mu\nu}(x)$$

where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

We automatically know that $F_{\mu\nu}(x)$ is locally gauge invariant because it was constructed out of the covariant derivative.

$$\boxed{S^{\mu\nu}F_{\mu\nu}}$$

$$\boxed{S^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}}$$

Case 2: Terms that contain ψ

Subcase 1: No derivatives

If you have ψ, then you have to have $\psi^\dagger$ somewhere in front of it in order to make a scalar by a dot product. You can't use a constant vector to form the dot product because that would destroy gauge invariance. We cannot have $(\psi^\dagger\psi)^2$ because ψ has mass dimension 3/2 and that would violate the renormalizability condition. We also cannot have $F_{\mu\nu}$ anywhere because it has mass dimension two, which would bring the total mass dimension of the fields above 4.

$$\boxed{S\psi^\dagger\psi}$$

$$\boxed{\psi^\dagger M\psi}$$

Subcase 2: One derivative

Applying the derivative to $\psi^\dagger$ is not fundamentally different because $\partial_\mu(\psi^\dagger)\psi = (\psi^\dagger\partial_\mu\psi)^*$.

$$\boxed{S^\mu \psi^\dagger D_\mu \psi}$$

$$\boxed{\psi^\dagger M^\mu D_\mu \psi}$$

Subcase 3: More than one derivative
There are no possible terms under this case because it violates the renormalizability condition

4 Lorentz Invariance

Before applying the Lorentz invariance constraint, the potential Lagrangian density looks like

$$\mathcal{L} = S_1^{\mu\nu} F_{\mu\nu} + S_2^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu} + S_3 \psi^\dagger \psi + \psi^\dagger M_1 \psi + S_4^\mu \psi^\dagger D_\mu \psi + \psi^\dagger M_2^\mu D_\mu \psi$$

After requiring Lorentz invariance we can eliminate the third and fifth terms and determine the matrix constants, at least up to a proportionality constant.

First we show that $\bar{\psi}\psi$ is Lorentz invariant, so that the matrix between $\psi^\dagger$ and ψ must be γ^0. The argument given in Sakurai's Advanced Quantum Mechanics section 3.4 goes as follows.

- Derive the Dirac equation based on the requirement that the operators are Lorentz invariant. This produces the gamma matrices to begin with.
- We then require the Dirac equation to be fully Lorentz invariant, so wave-function will have to have a particular transformation rule under Lorentz transformations so as to preserve the exact form of the Dirac equation. Let S be the operator defined by

  $$\psi'(x') = S\psi(x)$$

  Then

  $$\begin{aligned} \gamma^\mu \partial'_\mu \psi'(x') + m\psi'(x') &= 0 \\ \gamma^\mu {\Lambda_\mu}^\nu \partial_\nu (S\psi) + mS\psi &= 0 \\ S^{-1}\gamma^\mu S {\Lambda_\mu}^\nu \partial_\nu \psi + m\psi &= 0 \\ S^{-1}\gamma^\mu S {\Lambda_\mu}^\nu &= \gamma^\nu \\ S^{-1}\gamma^\mu S &= \gamma^\nu {\Lambda^\mu}_\nu \end{aligned}$$

  Note that S can be pulled out of the derivative because it corresponds to a Lorentz transformation, which acts in the same way at all points in spacetime.
- Use this constraint to find

  $$S_{rot}(\omega) = \cos\left(\frac{\omega}{2}\right) + i\sigma^{ij} \sin\left(\frac{\omega}{2}\right)$$

  $$S_{Lor}(\chi) = \cosh\left(\frac{\chi}{2}\right) - \sigma^{k0} \sinh\left(\frac{\chi}{2}\right)$$

  for a rotation by an angle ω or a boost by a velocity $\beta = \tan(\chi)$.
- Take the Hermitian conjugate of these expressions
- Obtain the relation

  $$S^\dagger = \gamma^0 S^{-1} \gamma^0$$

  using the fact that γ^0 commutes with σ^{ij} and anti-commutes with σ^{k0}.

- Calculate

$$\bar{\psi}'(x') = (\psi'(x'))^\dagger \gamma^0 = (S\psi(x))^\dagger \gamma^0 = \psi^\dagger(x) S^\dagger \gamma^0$$
$$= \psi^\dagger(x)\gamma^0 S^{-1}\gamma^0\gamma^0 = \bar{\psi}(x) S^{-1}$$

- Calculate

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x) S^{-1}\psi(x) = \bar{\psi}(x)\psi(x)$$

Therefore

$$\boxed{S_3 = 0 \quad \text{and} \quad M_1 \propto \gamma^0}$$

By construction, the Dirac equation is Lorentz invariant, so we know that we need to accompany ∂_μ with γ^μ (this is the non-rigorous sketch of the argument). Therefore

$$\boxed{S_4^\mu = 0 \quad \text{and} \quad M_2^\mu \propto \gamma^\mu}$$

As for the radiation terms, we know from general relativity that there are only two Lorentz invariant tensors, the metric and the fully anti-symmetric tensor. On page 24 of Carroll it says "A remarkable property of the above tensors–the metric, the inverse metric, the Kronecker delta, and the Levi-Civita symbol–is that, even though they all transform according to the tensor transformation law (1.63), their components remain unchanged in *any* inertial coordinate system in flat spacetime. ...In fact these are the *only* tensors with this property, although we won't prove it." So the second term splits into three cases

$$\boxed{S_2^{\alpha\beta\mu\nu} \propto g^{\alpha\mu} g^{\beta\nu} \quad \text{or} \quad S_2^{\alpha\beta\mu\nu} \propto g^{\alpha\beta} g^{\mu\nu} \quad \text{or} \quad S_2^{\alpha\beta\mu\nu} \propto \epsilon^{\alpha\beta\mu\nu}}$$

Note that any other permutation of the indices using just the metric will either be the same or the negative of one of these terms because $F_{\mu\nu}$ is anti-symmetric. The second option can be eliminated because it is the trace squared and anti-symmetric tensors have zero trace.

Peskin and Schroeder do not mention the first term. The argument that there are no possible terms is probably similar to the argument for the last case. If the $S_1^{\mu\nu}$ is the metric, then the term will be zero because $F^{\mu\nu}$ is traceless. Terms like $m^2\sigma^{\mu\nu}F_{\mu\nu}$ are not scalar because the $\sigma^{\mu\nu}$ are matrices.

5 P/T Invariance

After all cuts except for P/T invariance, there are only four possible terms, where the proportionality constants were inserted based on dimensional analysis and depend on the system of units used (in the case of the 1/4).

$$\mathcal{L}_4 = \bar{\psi}(i\not{D})\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - c\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}$$

The last term violates P and T invariance. When we drop it, we are left with exactly the QED Lagrangian.

6 Mini Theorem

If $[A, B]$ commutes with both A and B, then

$$e^A e^B = e^{A+B} e^{[A,B]/2}$$

7 Questions

- Prove whatever theorem is needed from general relativity to limit the number of possibilities for $S_2^{\alpha\beta\mu\nu}$. It is stated on page 24 of Carroll. What about $\sigma^{\mu\nu}$, does its components change?
- How do we derive the gauge transformation rule for ψ from the gauge transformations found in electromagnetism?
- Prove or disprove the rule that the only way to introduce A_μ and ∂_μ into the Lagrangian density is in the combination of the covariant derivative. This is probably more mathematical than most physicists care about, but it does seem reasonably important.
- What about terms with square roots? For example $m^2\sqrt{F_{\mu\nu}F^{\mu\nu}}$. That would be zero because the photon mass is zero, but we would have to include that case because it has a separate explanation for why it is not included. Zvi Bern says that this term is probably not renormalizable. The mass dimension counting heuristic is not a rigorous rule.

8 Annotated Bibliography

- Advanced Quantum Mechanics by Sakurai Sections 3.4 - Derives constraints of Lorentz invariance, Section 3.5 shows the Gordon decomposition of the vector current
- Quantum Field Theory by Peskin and Schroeder Section 15.1 - Derives covariant derivative and QED Lagrangian
- http://theory.physics.helsinki.fl/∼xfiles/qmii/04/ - Summary of Peskin and Schroeder Section 15.1 in the Lecture Notes
- Non-Relativistic Quantum Electrodynamics by Healy - Appendix D proves a theorem about multiplying exponential of non-commuting operators.
- Gauge Theories in Particle Physics by Aitchison and Hey - Page 154 explains that theories in which the coupling constants have negative mass dimension are not renormalizable.
- Spacetime and Geometry by Carroll - Page 24 states that the metric, inverse metric, Kronecker delta, and Levi-Civita symbol are the only tensors whose components are the same in all inertial coordinate systems.