1. *Quantum Mechanics* (Spring 2002)

The Hamiltonian of a one-dimensional harmonic oscillator in dimensionless units ($m = \hbar = \omega = 1$) is

$$H = a^\dagger + \frac{1}{2}$$

where $a = \frac{1}{\sqrt{2}}(x + ip)$, $a^\dagger = \frac{1}{\sqrt{2}}(x - ip)$. One of the unnormalized eigenfunctions of this Hamiltonian is given by the expression

$$\psi(x) = (2x^3 - 3x)e^{-x^2/2}$$

(a) What is the energy eigenvalue which corresponds to this wavefunction?

(b) Find the two other (unnormalized) energy eigenfunctions which are closest in energy to this wavefunction.

2. *Quantum Mechanics* (Spring 2002)

A system of two particles each with spin 1/2 is described by the Hamiltonian

$$H = A(S_{1z} + S_{2z}) + B\mathbf{S}_1 \cdot \mathbf{S}_2$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are the two spins, S_{1z} and S_{2z} are their z-components, and A and B are constants. Find all the energy levels of the Hamiltonian.

3. *Quantum Mechanics* (Spring 2002)

(a) Two Hermitian operators anticommute. Is it possible for them to have simultaneous eigenkets?

(b) Do position operators at unequal times commute in general in the Heisenberg representation? Give a simple example illustrating your answer.

(c) Explain how you would interpret the energy-time uncertainty relation. Illustrate your answer with a state that is a superposition of two energy eigenstates.

4. *Quantum Mechanics* (Spring 2002)

Consider a mass m particle in one dimension moving in the potential

$$V(x) = V_0 \left| \frac{x}{x_0} \right|$$

where V_0 and x_0 are constants. Estimate the ground state energy of the particle. Your score on this problem will be

$$\text{Your Score} = 20 \times e^{-\left(\frac{E-E_0}{E_0}\right)^2}$$

20 is the maximum score, E is your estimate, E_0 is the exact ground state energy, and E and E_0 are evaluated at $V_0 = \frac{\hbar^2}{m x_0^2} = 1\text{eV}$.

5. *Quantum Mechanics* (Spring 2002)

Charge on a circle: A small bead with charge e and mass m is confined to move on a circular ring in the x-y plane with radius r. A weak, uniform electric field of intensity E_0 pointing in the positive x direction is turned on.

(a) What are the eigenfunctions and energy eigenvalues for $E_0 = 0$? What are the degeneracies?

(b) For $E_0 \neq 0$, show that the electric field operator has vanishing matrix elements between degenerate eigenstates of the unperturbed Hamiltonian.

(c) Find an approximation for the energy levels which includes the first non-trivial term containing E_0.

6. *Statistical Mechanics and Thermodynamics* (Spring 2002)

Calculate the collision frequency for the collisions between the molecules of a gas and a fixed sphere of diameter D. The molecules have an average diameter d. The gas has a temperature T.

7. *Statistical Mechanics and Thermodynamics* (Spring 2002)

This is an essay question. Answer two of the following three questions.

(a) You are asked about the second law of thermodynamics, and you give one of the formulations, that there is no process the sole effect of which is the conversion of heat into work. The inquirer then points out that a steam engine converts heat into work. Explain how this is not a violation of the second law of thermodynamics. Your explanation should include an analysis of the steam engine, and a discussion of heat engines in general.

(b) You read an article in a physics journal in which a group of researchers announce that they have cooled a system to absolute zero. Discuss why one ought to be skeptical of this claim. Invoke the appropriate laws of thermodynamics.

(c) Explain, using the laws of thermodynamics, why a substance cannot have a negative heat capacity.

8. *Electricity and Magnetism* (Spring 2002)

Consider the AC current density in a conductor obeying Ohm's law, $\mathbf{J} = \sigma\mathbf{E}$, where σ is the constant real conductivity. Suppose that we have a conductor which is a $2D$ metal sheet of thickness d located in the x-z plane and that current flows in the x-direction. If the AC current density is given by the expression $J_x(y,t) = \mathrm{Re}(\mathcal{J}_x(y)e^{i\omega t})$,

(a) Find $\mathcal{J}_x(y)$.

(b) Plot $|\mathcal{J}_x(y)|$ v.s. y.

(c) Find the phase shift of the current density between the center and the edge for $\omega\mu_0\sigma d \gg 1$.

9. *Electricity and Magnetism Lorentz transformation in one spatial dimension.*

(a) Assuming the x and t transform according to the Lorentz transformation law, show that the combination $(ds)^2 = (dx)^2 - c^2(dt)^2$ is the same in all inertial frames.

(b) Show that the elapsed time $d\tau$ between two events ocurring at the same location in the laboratory is related to the elapsed time dt' in the frame M moving along the $+x$ axis with speed v according to

$$d\tau = dt'\sqrt{1 - \frac{1}{c^2}\left(\frac{dx'}{dt'}\right)^2} = dt'/\gamma.$$

Here $dx'/dt' = -v$ is the velocity of the fixed laboratory point as seen in the frame M and $\gamma = 1/\sqrt{1-\beta^2}$, where $\beta = v/c$.

(c) Show that the spatial separation ds between two events occurring simultaneously ($dt = 0$) in the laboratory is related to the spatial separation dx' in the moving frame M by

$$ds = dx'\sqrt{1 - c^2\left(\frac{dt'}{dx'}\right)^2} = A dx'$$

Here, dt' is the time in the moving frame which elapses between the two events. Determine the proportionality constant A.

(d) The ratio dx'/dt' is a function of the relative speed v between the two frames. Is it the same function in parts (b) and (c)? Explain.

10. *Electricity and Magnetism* (Spring 2002)

An infinitely long solenoid of circular cross-section of radius a carries a current I along helical windings of n turns per axial length. The current is closed via a straight conductor of radius $b < a$ centered along the axis.

(a) Find the magnetic field inside and outside the solenoid.

(b) Calculate the self-inductance per axial length due to fields produced the the solenoidal and axial currents.

(c) Find the "inner" inductance due to magnetic fields inside the conductors.

11. *Electricity and Magnetism* (Spring 2002)

Forces on conductors

(a) A grounded spherical conductor of radius a is placed at a distance $r \gg a$ far from a point charge q. Determine the direction and functional form of any force experienced by the conductor due to the presence of the point charge. (You don't need to calculate and proportionality constant.)

(b) What is the direction and functional form of any force experienced by the conductor in part (a) if the conductor is isolated and uncharged and has an unknown shape, but the same small volume? If your answer is different from that in part (a), explain the difference in physical terms. (Hint: Use a multipole expansion of the force acting on an arbitrary charge distribution in an external electric field.)

12. *Electricity and Magnetism* (Spring 2002)

Consider an infinitely long, straight wire along the $\hat{\mathbf{z}}$-axis with a capacitor at $z = 0$, as shown in the picture below. The wire carries current I. Suppose that the plates of the capacitor are circular of radius r_c and are separated by a distance d; there is vacuum between them.

(a) What is the capacitance of the capacitor? Assume that $r_c \gg d$ so that you can ignore fringing fields.

(b) What is the magnetic field at $z = 0$ at a distance $r \gg r_c$ from the capacitor?

(c) What is the magnetic field a distance r from the wire for $|z| \gg d$?

13. *Statistical Mechanics and Thermodynamics* (Spring 2002)

Consider an ideal monoatomic gas in which each atom has two internal energy states, one an energy Δ above the other. There are N atoms in a volume V at temperature T.

Find the (a) chemical potential, (b) free energy, (c) entropy, (d) pressure and (e) heat capacity at constant pressure.

14. *Statistical Mechanics and Thermodynamics* (Spring 2002)

(a) What is the free energy (as a function of temperature, T, volume, V, and particle number, N) of an ideal gas obeying Maxwell-Boltzmann statistics?

(b) Assume that the ideal gas is made up of hydrogen atoms. Now the free energy must include a contribution reflecting the different possible electronic excited states of the hydrogen atoms. Show that this contribution diverges. What cuts off this divergence in a real gas?