1. *Quantum Mechanics* (Spring 2004)

The table below shows some Clebsch-Gordan coefficients. If two particles have spin 1/2 and 3/2 respectively, write down all composite states $|sm\rangle$ in terms of the uncoupled states using Dirac notation. You may use the following table if you wish. (A square root is understood for all entries in the table below, with the $\pm$ sign outside the radical.)

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
⋮	⋮			

3/2 × 1/2

m_1	m_2	2, +2
+3/2	+1/2	1

m_1	m_2	2, +1	1, +1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	2, 0	1, 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	2, −1	1, −1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	2, −2
−3/2	−1/2	1

2. *Quantum Mechanics* (Spring 2004)

A hydrogen atom is in the ground state ($n = 1, l = m = 0$) for $t < 0$. Suppose the atom is placed between the plates of a capacitor, and a weak, spatially uniform but time-dependent decaying field is applied at $t = 0$. The field (for $t > 0$) is

$$\mathbf{E} = \mathbf{E}_o e^{-\gamma t}$$

for some $\gamma > 0$. Take $\mathbf{E}_o$ along the z-axis. What is the probability (to first order in E_o) that the atom will be in each of the four $n = 2$ states as $t \to \infty$? Neglect spin.

You may need some of the functions $R_{nl}(r)$ and $Y_l^m(\theta, \phi)$ in the following table:

$$a^{\frac{3}{2}} R_{10}(r) = 2e^{-r/a} \qquad a^{\frac{3}{2}} R_{20}(r) = \frac{1}{\sqrt{2}}\left(1 - \frac{r}{2a}\right) e^{-r/2a} \qquad a^{\frac{3}{2}} R_{21}(r) = \frac{1}{2\sqrt{6}}\frac{r}{a} e^{-r/2a}$$

$$Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \qquad Y_1^0(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos(\theta) \qquad Y_1^{\pm 1}(\theta, \phi) = \mp\sqrt{\frac{3}{8\pi}} \sin(\theta) e^{\pm i\phi}$$

Table 1: Some hydrogen atom radial wave functions and spherical harmonics. a is the Bohr radius: $a = \hbar/mc\alpha$.

And an integral

$$\int_0^\infty x^n e^{-x/a}\, dx = a^{n+1} n!$$

3. *Quantum Mechanics* (Spring 2004)

The normalized wave function of a one-dimensional particle is

$$\psi(x) = Ne^{-\kappa x^2/2}$$

for some $\kappa > 0$. N is real and positive.

(a) What is N?

(b) What is the expectation value of x^2?

(c) What is the momentum space wave function $\langle p|\psi\rangle$?

(d) What is the expectation value of p^2?

(e) The Hamiltonian is

$$H = \frac{p^2}{2m} + V(x)$$

What is the potential $V(x)$?

4. *Quantum Mechanics* (Spring 2004)

The electron neutrino $|\nu_e\rangle$ and the muon neutrino $|\nu_\mu\rangle$ are the possible neutrino states produced and detected in experiments, but they are not necessarily eigenstates of the Hamiltonian. Rather, if the state is known to have momentum p, it is some linear combination of the energy eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ of the form

$$|\nu_e\rangle = \cos(\theta)\,|\nu_1\rangle + \sin(\theta)\,|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin(\theta)\,|\nu_1\rangle + \cos(\theta)\,|\nu_2\rangle$$

where

$$H\,|\nu_1\rangle = \sqrt{p^2c^2 + m_1^2c^4}\,|\nu_1\rangle$$

$$H\,|\nu_2\rangle = \sqrt{p^2c^2 + m_2^2c^4}\,|\nu_2\rangle$$

for two possibly different masses m_1 and m_2, and some "mixing angle" θ. If it is known that a neutrino was definitely a ν_μ when it was produced, what is the probability of detecting a ν_e after it has traveled a distance L? Assume that $m_1c \ll p$ and $m_2c \ll p$, so that the neutrinos are moving at almost (or even exactly) the speed of light, (so you can ignore corrections of the order $1 - v/c$ compared to terms of order 1) and state your result to first order in the difference $\Delta m^2 = m_1^2 - m_2^2$.

This is a simplified version of an actual neutrino oscillation experiment like the super-Kamiokande detector experiment a few years ago. In reality there is a third neutrino $|\nu_\tau\rangle$.

5. *Quantum Mechanics* (Spring 2005)

Calculate the transmission coefficient for a particle of energy $E > 0$ scattering off the $1D$ potential well $V(x) = V_0$ for $0 < x < a$, $V(x) = 0$ elsewhere, $V_0 < 0$. Are there resonance phenomena?

6. *Statistical Mechanics and Thermodynamics* (Spring 2004)

Consider a gas of relativistic, conserved bosons. The relation between energy and momentum is

$$E = |\mathbf{p}|c$$

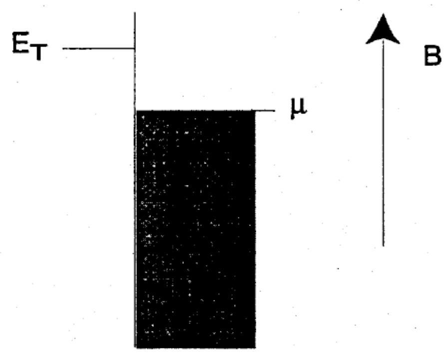
(a) Derive the condition for Bose-Einstein condensation in three dimensions.

(b) Does Bose-Einstein condensation occur in two dimensions? Justify your answer.

(c) What is the highest dimension for which Bose-Einstein condensation does not occur?

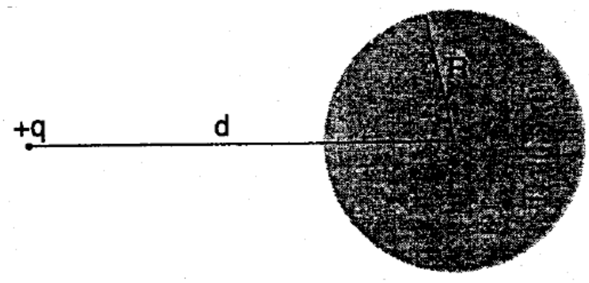
7. *Statistical Mechanics and Thermodynamics* (Spring 2004)

A quantum state at energy E_T is embedded in a system with a degenerate Fermi gas as, for instance, occurs with an impurity state with energy E_T in a degenerate semiconductor with a sea of conducting electrons at chemical potential μ. You may assume that $E_T > \mu$. The impurity, which has a spin of 1/2, can take an additional electron from the large bath of electrons (costs Coulomb energy U), to form a spin-singlet state. For a given temperature T and magnetic field B, calculate the ratio of the probability for the trap being empty to that for the trap being filled by an additional electron.



8. *Electricity and Magnetism* (Spring 2004)

A point charge q is located a distance d from the center of a conducting sphere of radius R. What must the total charge on the conducting sphere be for the force on the point charge to be zero?

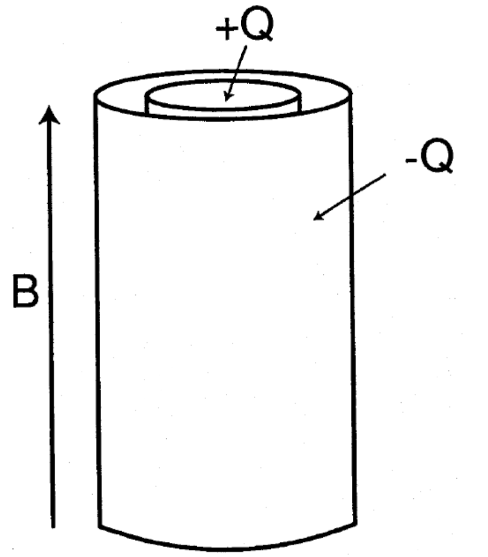


9. *Electricity and Magnetism* (Spring 2004)

Consider the infinite two-dimensional conducting plane depicted in the figure. The right half is maintained at electrostatic potential V_0 while the left half is maintained at potential $-V_0$. What is the potential above the plane?

10. *Electricity and Magnetism* (Spring 2004)

Consider a cylindrical capacitor of length L with charge $+Q$ on the inner cylinder of radius a and $-Q$ on the outer cylindrical shell of radius b. The capacitor is filled with a lossless dielectric with dielectric constant equal to 1. The capacitor is located in a region with a uniform magnetic field B, which points along the symmetry axis of the cylindrical capacitor. A flaw develops in the dielectric insulator, and a current flow develops between the two plates of the capacitor. Because of the magnetic field, this current flow results in a torque on the capacitor, which begins to rotate. After the capacitor is fully discharged (total charge on both plates is zero), what is the magnitude and direction of the angular velocity of the capacitor? The moment of inertia of the capacitor (about the axis of symmetry) is I, and you may ignore fringing fields in the calculation.



11. *Electricity and Magnetism* (Spring 2004)

Consider a plasma of free charges of mass m and charge e at constant density n. What is the index of refraction for electromagnetic waves of frequency ω which are incident upon this plasma?

12. *Electricity and Magnetism* (Spring 2004)

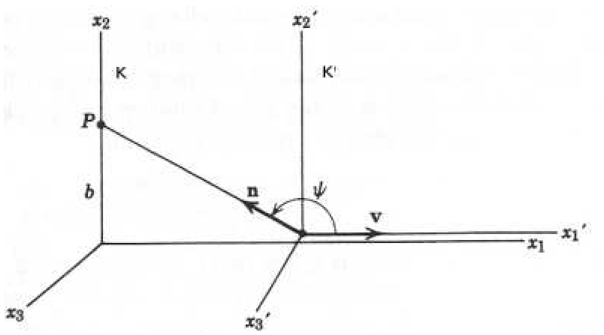
The fields due to a charge in motion are:

$$\mathbf{E}(\mathbf{x},t) = e\left[\frac{\mathbf{n}-\overrightarrow{\beta}}{\gamma^2(1-\overrightarrow{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret} + \frac{e}{c}\left[\frac{\mathbf{n}\times\left[\left(\mathbf{n}-\overrightarrow{\beta}\right)\times\dot{\overrightarrow{\beta}}\right]}{(1-\overrightarrow{\beta}\cdot\mathbf{n})^3R}\right]_{ret}$$

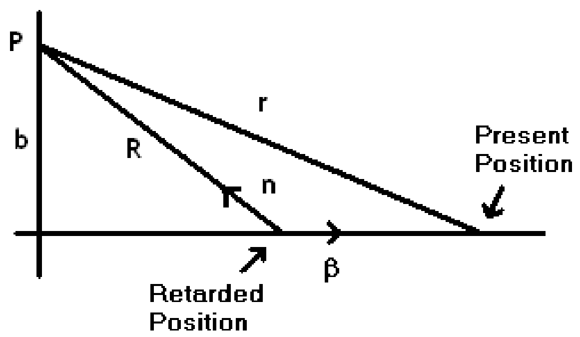
$$\overrightarrow{B}(\mathbf{x},t) = [\mathbf{n}\times\mathbf{E}]_{ret} \tag{1}$$

where $\overrightarrow{\beta} = \mathbf{v}/c$, $\mathbf{n}$ is a unit vector in the direction of the observation point $\mathbf{x}$, $\gamma = 1/\sqrt{1-\beta^2}$ and 'ret' means the quantities are evaluated at the retarded time (so e.g. $\mathbf{n}$ in (1) is the unit vector pointing from the retarded position of the charge to the observation point).

(a) Identify in the expression (1) 'static fields' and 'radiation fields'. Show how the static field part can be obtained from a Lorentz transformation of the fields of a static charge.



Hint: You may want to refer to the sketch above, where K' is the rest frame of the particle and P the observation point (which the particle passes at impact parameter b); suppose K and K' coincide at $t = t' = 0$. Write the fields in K', transform to the K coordinates, then transform the fields to K. Now you have the fields of the moving charge in terms of its present position. Show that the parallel and transverse components of E are the same as given in (1) in terms of the retarded position. The sketch below may be useful, where R is the retarded distance and r the present distance. You have to express $R^2(1-\mathbf{b}\cdot\mathbf{n})^2$ in terms of r and b etc.



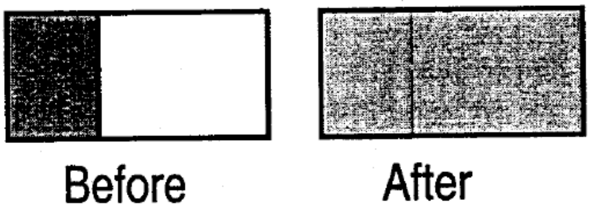
(b) Using the radiation field part of (1) in the non-relativistic limit ($\beta \ll 1$), calculate the average power radiated per unit solid angle by a charge q oscillating along the z-axis: $z(t) = A\cos(\omega t)$, where z is the position of the charge. The power is a function of the azimuthal angle θ, and 'average' means average in time (i.e. average over 1 oscillation).

13. *Statistical Mechanics and Thermodynamics* (Spring 2004)

A van der Waals gas has the following equation of state:

$$P(T, V) = \frac{NkT}{(V - bN)} - a\left(\frac{N}{V}\right)^2$$

This gas is held in a container of negligible mass which is isolated from its surroundings. The gas is initially confined to 1/3 of the total volume of the container by a partition (a vacuum exists in the other 2/3 of the volume). The gas is initially in thermal equilibrium with temperature T_i. A hole is then opened in the partition, allowing the gas to irreversibly expand to fill the entire volume (V). What is the new temperature of the gas after thermal equilibrium has been re-established? (Hint: Note that the specific heat at constant volume for a van der Waals gas is the same as that for an ideal gas.)



14. *Statistical Mechanics and Thermodynamics* (Spring 2004)

Imagine that the sites of a lattice are occupied with probability p and are unoccupied with probability $1 - p$. If two neighboring sites are occupied, then we consider them to be part of the same cluster. As p is increased, larger clusters become more likely. When $p > p_c$ for some p_c (the 'percolation threshold') which depends on the dimension and the particular lattice, there will be a cluster which extends all the way across the system. For $p < p_c$, we will call the mean cluster size S.

(a) What is the percolation threshold, p_c, of a one-dimensional chain?

(b) In an infinite one-dimensional chain, what is the probability n_s that a given site is the left end of a cluster of length precisely s (in terms of p and s)?

(c) $n_s s$ is the probability that a given site is on a cluster (anywhere, not just the left end) of length s. p is the probability that a given site is on a cluster of any non-zero size. What is the mean cluster size, S, in terms of $n_s s$ ($s = 1, 2, ...$) and p?

(d) Using your results from parts (b) and (c), what is the mean cluster size, S, of a one-dimensional chain as a function of p alone?