1. *Quantum Mechanics* (Spring 2005)

Consider a particle of charge q in a one-dimensional harmonic oscillator potential. Suppose there is also a weak electric field E so that the potential is shifted by

$$H' = -qEx$$

(a) Calculate the correction to the simple harmonic oscillator energy levels through second order in perturbation theory.

(b) Now solve the problem exactly. How do the exact energy levels compare with the perturbative result in (a)?

2. *Quantum Mechanics* (Spring 2005)

Show that in one space dimension any attractive potential, no matter how weak, always has at least one bound state. *Hint:* Use the variational principle with some appropriate trial wave function such as the normalized Gaussian

$$\psi(x) = \left(\frac{2b}{\pi}\right)^{1/4} e^{-bx^2}$$

where b is a parameter.

3. *Quantum Mechanics* (Spring 2005)

A beam of particles scatters off an impenetrable sphere of radius a. That is, the potential is zero outside the sphere, and infinite inside. The wave function must therefore vanish at $r = a$.

(a) What is the S-wave ($l = 0$) phase shift as a function of the incident energy or momentum?

(b) What is the total cross section in the limit of zero incident kinetic energy?

4. *Quantum Mechanics* (Spring 2005)

An electron is at rest in a constant magnetic field pointing along the z direction. The Hamiltonian is

$$H = -\mu \cdot \mathbf{B} = g\mu_o \frac{\mathbf{s}}{\hbar} \cdot \mathbf{B}$$

where $\mathbf{B} = B_o \hat{n}_z$. $\mathbf{s}$ is the electron spin. Since the electron is at rest, you can treat this as a two-state system. Let $|\psi_\pm\rangle$ be the eigenstates of s_z with eigenvalues $\pm\frac{\hbar}{2}$ respectively.

(a) What are the eigenstates of the Hamiltonian, and what is the energy difference between them?

(b) At time $t = 0$ the electron is in an eigenstate of s_x with eigenvalue $+\hbar/2$. Calculate $|\psi(t)\rangle$ for any t.

(c) For the state you calculated in part (b), what are the expectation values of the three components of the spin at any time t?

5. *Quantum Mechanics* (Spring 2005)

An electron moves in a hydrogen atom potential – ignoring spin and relativity – in a state $|\psi\rangle$ that has the wave function

$$\psi(r,\theta,\phi) = NR_{21}(r)\left[2iY_1^{-1}(\theta,\phi) + (2+i)Y_1^0(\theta,\phi) + 3iY_1^1(\theta,\phi)\right]$$

where the $Y_l^m(\theta,\phi)$ are the spherical harmonics, $R_{nl}(r)$ are the normalized hydrogen atom wave functions, and N is a positive real number.

(a) Calculate N.

(b) What is the expectation value of L_z? ($\hbar\mathbf{L} = \mathbf{r}\times\mathbf{p}$)

(c) What is the expectation value of $\mathbf{L}^2$?

(d) What is the expectation value of the kinetic energy in terms of $\hbar, c$, the electron charge e or the fine-structure constant α, and the electron mass m?

Note: The explicit forms of the functions that appear in $\psi(r,\theta,\phi)$ above are

$$R_{21}(r) = \frac{1}{2\sqrt{6}}\frac{r}{a^{5/2}}e^{-r/2a} \qquad Y_1^{\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin(\theta)e^{\pm i\phi} \qquad Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos(\theta)$$

6. *Statistical Mechanics and Thermodynamics* (Spring 2005)

A closed container is divided by a wall into two equal parts (A and B), each of volume $V/2$. Part A contains an ideal gas with $N/2$ molecules of mass M_1 while part B contains an ideal gas with $N/2$ molecules of mass M_2. The container is kept at a fixed temperature T. The molecules of each kind are all identical, but distinguishable from the molecules of the other kind.

(a) The partition function $Z(N)$ of an ideal gas of N particles of mass M in a volume V is given by

$$Z(N) = \frac{1}{N!}\left(\frac{V}{\sqrt{2\pi\hbar^2/Mk_BT}}\right)^N$$

Give the partition function of the gas in the container before and after the wall is removed. What are the entropy and pressure before and after the wall is removed?

(b) How much heat is absorbed or released following the removal of the wall? Is the removal of the wall a reversible or irreversible process?

(c) Same question as (b), but now for the case that the two kinds of molecules are indistinguishable from each other (so $M_1 = M_2$). Compare your answers for (b) and (c) and provide a physical explanation for the difference in entropy between the two cases.

7. *Statistical Mechanics and Thermodynamics* (Spring 2005)

A (nearly) ideal gas with a temperature T and pressure P contains atoms of mass M that are either in the ground state or in the first excited state. An atom that returns to the ground state from the first excited state emits a photon of frequency f_o. For a stationary observer observing the spectral line emitted by a *moving* atom, this frequency is shifted by the Doppler effect to

$$f(v_{\|}) = f_o(1 + v_{\|}/c)$$

where c is the velocity of light and $v_{\|}$ is the projection of the velocity of the atom on the line of sight from the observer to the atom.

(a) What is the statistical distribution $P(f)$ of the frequency of the spectal line? Assume the atoms obey the Maxwell-Boltzmann distribution.

(b) Obtain from $P(f)$ the contribution by the Doppler effect to the width $\sqrt{\langle (f - f_o)^2 \rangle}$ of the spectral line. Can you think of a way this effect could be exploited in the study of stellar atmospheres?

(c) The *natural* line shape $P(f)$ of an atomic spectral line is, according to quantum mechanics, given by

$$P(f) \sim \frac{1}{(f - f_o)^2 + \tau^{-2}}$$

where τ is the *lifetime* of the excited state. For atoms in a dense gas, the actual lifetime of the excited state is not intrinsic, but instead determined by the time interval between successive collisions between atoms. Let the cross section of an atom equal σ. Obtain an expression for τ in terms of σ, the pressure P and the temperature T. Under which conditions will this "collisional" broadening of the spectral line dominate over the Doppler broadening as computed under (b)?

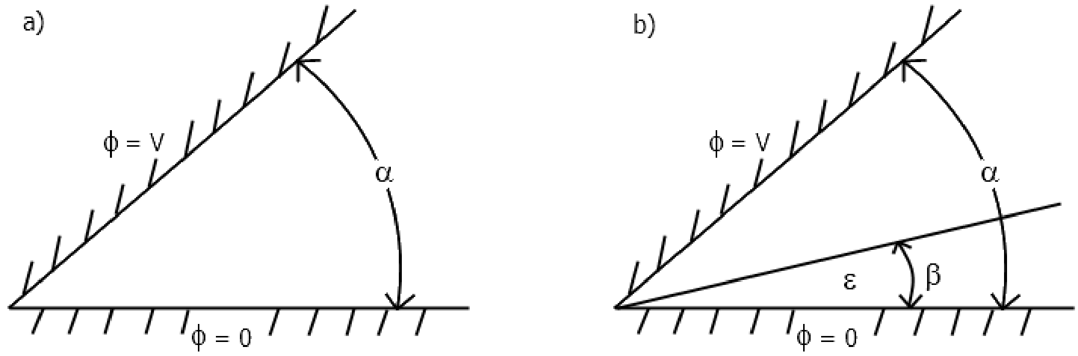
8. *Electricity and Magnetism* (Spring 2005)

Consider a two-dimensional (r, θ) electrostatic problem consisting of two infinite plates making an angle α with each other and held at a potential difference V, as shown below:

(a) Find the potential $\phi(r, \theta)$ in the vacuum region between the plates.

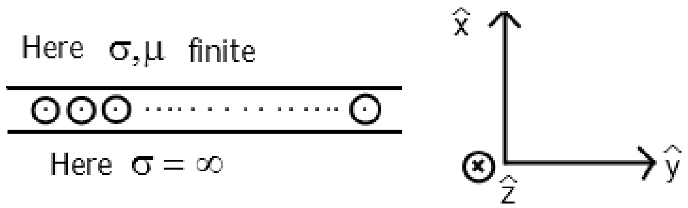
Now insert a wedge dielectric, of dielectric coefficient ϵ, and angle β, resting on the bottom plate as shown below:

(b) Find the pressure experienced by the bottom plate at a distance r from the apex (from the line joining the two plates).



9. *Electricity and Magnetism* (Spring 2005)

An infinitely thin current sheet carrying a surface current $\lambda = \lambda_o \hat{z}\cos(\omega t)$ is sandwiched between a perfect conductor ($\sigma = \infty$) and a material having finite conductivity σ and magnetic permeability μ. The angular frequency ω is sufficiently low that magnetostatic conditions prevail. λ_o is a constant, $\hat{z}$ is a unit vector parallel to the interface located at $x = 0$, and t is the time.



(a) Find the appropriate partial differential equation that governs the behavior of the magnetic field $\mathbf{H}$ for $x > 0$ (above the current sheet). Do not solve.

(b) What is the appropriate boundary condition for $\mathbf{H}$ in this system?

(c) Find the magnetic field $\mathbf{H}$ at an arbitrary distance $x > 0$ at time t.

10. *Electricity and Magnetism* (Spring 2005)

A relativistic charged particle of charge q and rest-mass m_o is in a region of uniform magnetic field $B_o\hat{z}$. At time $t = 0$ the particle has zero velocity along $\hat{z}$ (that is $\beta_z = v_z/c = 0$) and finite transverse speed $\beta_\perp = \beta_o$, with

$$\beta_\perp = \sqrt{v_x^2 + v_y^2}/c$$

Here, x, y, and z are Cartesian coordinates in the lab frame.

(a) What is the value of $\beta_\perp(t)$ for $t > 0$?

(b) What is the angular frequency Ω of rotation (that is, the gyrofrequency)? No need for a calculation, just identify Ω.

(c) Now apply a uniform electric field $E_o\hat{z}$, parallel to $\mathbf{B}$, starting at $t = 0$. Without solving the detailed equations, conclude what happens to the $\beta_\perp$ in part (a). Does it change?

11. *Electricity and Magnetism* (Spring 2005)

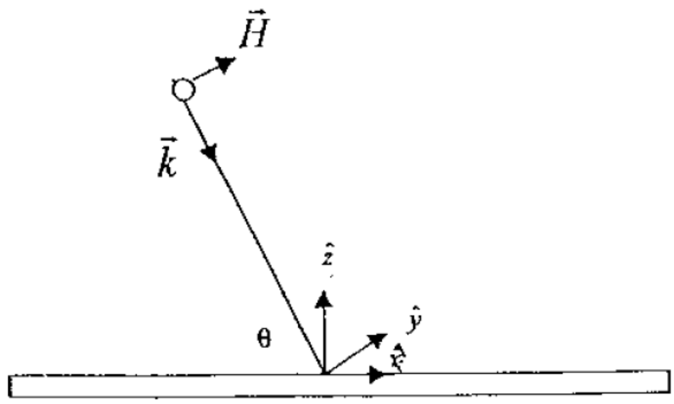
A linearly polarized electromagnetic wave propagating through the vacuum falls on a flat metallic surface. The wavelength of the incident wave is λ. The angle between the wave vector $\mathbf{k}$ and the metal surface is equal to θ. The electric field has a magnitude E_o and a direction normal to the page (positive y direction, see Figure). Assume that the metal surface has infinite conductivity.

(a) Show that the boundary conditions can be obeyed by adding a *reflected* wave to the incident plane wave. Draw, in the Figure, the directions of the electric and magnetic field vectors of the reflected wave such that the boundary conditions hold at the surface.

(b) Calculate the *time-averaged* Poynting vector of the incident plus the reflected wave in terms of E_o. Along what direction is the electromagnetic energy being transported by the two waves?

(c) Show that the repeat length of the interference pattern of the two waves along the surface of the plates is given by $\lambda/\cos(\theta)$, while the repeat length perpendicular to the surface is given by $\lambda/\sin(\theta)$. It follows from this that one can insert a *second* metal plate at a height $D(m) = m(\lambda/2)\sin(\theta)$ above the first metal plate, with m an integer, without perturbing the wave pattern.

(d) Using (c) compute the *phase velocity* $v(f)$ of an electromagnetic wave trapped between two parallel plates with spacing D as a function of the frequency f of the wave. This phase velocity should *diverge* as you reduce f. Demonstrate that the fact that $v(f)$ exceeds the velocity of light for some f is not a violation of the principle of special relativity (even though $v > c$ for small f).



12. *Electricity and Magnetism* (Spring 2005)

A thin copper ring (conductivity σ, density ρ) is suspended so it can rotate freely about one diameter. There is a uniform magnetic field $\mathbf{B}$ perpendicular to the axis of rotation. At time $t = 0$ the ring is set rotating with frequency ω_o. Calculate the time it takes the frequency to decrease to $1/e$ of its original value, assuming the energy goes into Joule heating.

13. *Statistical Mechanics and Thermodynamics* (Spring 2005)

Consider the one-dimensional Ising model on a periodic lattice, that is, a chain of N spins, with sping $s_i = \pm 1$ residing on the i-th site, $i = 1, ..., N$, forming a closed loop. The partition function in the presence of an external magnetic field H is then

$$Z_N = \sum_{\{s_i = \pm 1\}} \exp\left(\beta J \sum_{i=1}^{N} s_i s_{i+1} + \beta H \sum_{i=1}^{N} s_i\right)$$

where $\beta = 1/kT$. Define the 2×2 transfer matrix $\mathbf{T}$ with elements

$$T(s, s') = \exp\left[\nu s s' + \frac{B}{2}(s + s')\right] \qquad (s, s' = \pm 1)$$

where we let $\nu = \beta J$ and $B = \beta H$.

(a) Show that

$$Z_N = \mathrm{Tr}(\mathbf{T}^N)$$

and hence

$$Z_N = \lambda_1^N + \lambda_2^N$$

where λ_1 and λ_2 are the two eigenvalues of $\mathbf{T}$.

(b) Determine λ_1, λ_2. If λ_1 denotes the larger eigenvalue, observe that λ_2/λ_1 is strictly less than one for all $\nu > 0$. Hence show that the free energy per spin in the thermodynamic limit $N \to \infty$ is given by

$$-F/kT = \ln(\lambda_1)$$

(c) What is the spontaneous magnetization per spin for any $\nu > 0$?

14. *Statistical Mechanics and Thermodynamics* (Spring 2005)

A photon gas in thermal equilibrium is contained within a box of volume V at temperature T.

(a) Use the partition function to find the average number of photons $\bar{n}_r$ in the state having energy E_r.

(b) Find a relationship between the radiation pressure P and the energy density u (i.e. the average energy per unit volume).

(c) If the volume containing the photon gas is decreased adiabatically by a factor of 8, what is the final pressure if the initial pressure is P_o?