1 Partial Differential Equations

Linear Second-Order PDEs:
- Laplace Eqn (elliptic PDE)
- Poisson Eqn (elliptic PDE)
- Helmholtz Eqn (elliptic PDE)
- Wave Eqn (hyperbolic PDE)

2 Laplace Equation: $\nabla^2 u = 0$

2.1 Relevance

- Electrostatics: Find $\Phi$ and/or $E$ or $D$ in a region where $\rho = 0$.
- Magnetostatics: Find $\Phi_m$ and/or $B$ or $H$ when $J = 0$ everywhere. (For this example, the permeability should be linear, isotropic, and homogeneous (LIH), or piecewise constant.)

2.2 The Equation and Solution Methods

The Laplace equation has a general, abstract representation (shown in Equation 10) that uses the Laplace operator $\nabla^2$ (or $\Delta$) and that may take different explicit forms when expressed in different coordinate systems and in different numbers of dimensions. Its meaning is derived from the meanings of the gradient and divergence, which are defined like the derivative using limiting procedures that are appropriate for the given dimensions. Laplace’s equation is the homogeneous form of the Poisson equation; the solutions are functions in the kernel of the Laplace operator. A solution that is twice continuously differentiable is called a harmonic function. More general solutions may be formed as (possibly infinite) sums of harmonic functions.

To solve this equation, you start by picking a coordinate system that matches the geometry of the given boundary conditions (BCs). It seems that from here we always use the procedure of separation of variables, and it is generally convenient to use special functions liberally (otherwise you can use more manual methods like starting with an arbitrary power series function and deriving the special functions and their recurrent coefficient relations yourself). Once we find a particular solution that matches the given boundary conditions, we know we’ve found our unique solution and we’re done. Of course, the BCs provide the real challenge in our pursuit of solutions. We may have Dirichlet BCs, where $u$ is specified (as some value or function) on the boundary, or we may have Neumann BCs, where the derivative of $u$ is specified, but we don’t usually deal with Robin BCs (a kind of combination of Dirichlet and Neumann) or mixed BCs.

\footnote{See the Wikipedia article on Partial Differential Equations to learn about the complications in the issue of uniqueness of solutions.}
2.3 Example Solution Derivations

Let’s begin by solving the Laplace equation in 2D Cartesian coordinates for some potential $\Phi$: $\nabla^2 \Phi = (\partial_x^2 + \partial_y^2) \Phi = 0$. First of all note that we would only be selecting this form of equation if there was a rectilinear geometry in our boundary conditions; for example, say that you have a long rectangular tube where you know the potential on the surface of the tube, but the potential doesn’t change along the length of the tube. We could put the $z$-axis along one corner of the tube and say that the sides are located at $x = 0$, $y = 0$, $x = a$, and $y = b$, with the potential given as $V(0, y)$ and $V(a, y)$ (arbitrary functions), $V(x, 0) = 0$, and $V(x, b) = (5 V)$.

We will initially start out with 3D Cartesian coordinates, but the boundary conditions will simplify the situation and bring us into 2D. Since $\Phi$ must match the $z$-independent boundaries, we have $\Phi(x, y, z) = \Phi(x, y)$ and for each harmonic that we use (where I denote harmonics with a subscript $h$), $\Phi_h(x, y) = X(x) Y(y)$. Thus, our full solution will be $\Phi(x, y) = \sum \Phi_h(x, y)$, where each harmonic satisfies the equation. We will see that we have to be strategic about which harmonics we use, however. First, though, let’s start with the method of separation of variables:

$$0 = \nabla^2 \Phi_h$$

(1)

(2)

(3)

(4)

(5)

(6)

(7)

Then we divide by $\Phi_h = X Y$:

$$0 = \frac{1}{X(x)} d_x^2 X(x) + \frac{1}{Y(y)} d_y^2 Y(y)$$

(8)

and since each function, $X$ and $Y$, is independent, we have two terms here that are independent but add up to a constant (zero). That means that each term must be constant: let’s define an arbitrary positive constant $k$ so that one term equals $k^2$ and the other equals $-k^2$. To be thorough, we should observe both cases, where each term could be positive or negative.

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_x^2 X_k(x) = k^2 X_k(x)$</td>
<td>$d_x^2 X_k(x) = -k^2 X_k(x)$</td>
</tr>
<tr>
<td>$d_y^2 Y_k(y) = -k^2 Y_k(y)$</td>
<td>$d_y^2 Y_k(y) = k^2 Y_k(y)$</td>
</tr>
</tbody>
</table>

When solving, we will concentrate on one case at a time. For each case, we note that the solutions for $X_k$ and $Y_k$ come in special pairs. If a term is positive, that will allow us to select from the harmonic functions $e^{\pm kx}$, $\sinh(\pm kx)$, and $\cosh(\pm kx)$, or $\sin(\pm ikx)$ and $\cos(\pm ikx)$, for instance. If a term is negative, that will allow us to select from the harmonics $\sin(\pm kx)$ and $\cos(kx)$, or $e^{\pm ikx}$, $\sinh(\pm ikx)$, and $\cosh(ikx)$. Of course we may also select from (If we allowed $k^2$ to be a complex number, then... things would be more complex...)

(9)

2.4 Equations and Solutions

$$\nabla^2 u \equiv \nabla \cdot \nabla u = 0$$

(10)
2D Cartesian Coords, \( u = u(x, y) \):

\[
\left[ \partial_x^2 + \partial_y^2 \right] u = 0 \tag{11}
\]

\[
u(x, y) = \sum_i (A_i x + C_i \cos b_i x + )() \tag{12}
\]

3D Cartesian Coords, \( u = u(x, y, z) \):

\[
\left[ \partial_x^2 + \partial_y^2 + \partial_z^2 \right] u = 0 \tag{13}
\]

2D Polar Coords, \( u = u(s, \phi) \):

\[
\left[ \frac{1}{s} \partial_s (s \partial_s) + \frac{1}{s^2} \partial\phi^2 \right] u = 0 \tag{14}
\]

\[
\partial_s^2 + \frac{1}{s} \partial_s + \frac{1}{s^2} \partial\phi^2 u = 0 \tag{15}
\]

\[
u(s, \phi) = (a_0 + b_0 \ln s)(A_0+) + \sum_{n=1}^{\infty} (a_n s^n + b_n s^{-n})(A_n \cos n\phi + B_n \sin n\phi) \tag{16}
\]

3D Cylindrical Coords, \( u = u(s, \phi, z) \):

\[
\left[ \frac{1}{s} \partial_s (s \partial_s) + \frac{1}{s^2} \partial\phi^2 + \partial_z^2 \right] u = 0 \tag{17}
\]

\[
\partial_s^2 + \frac{1}{s} \partial_s + \frac{1}{s^2} \partial\phi^2 + \partial_z^2 u = 0 \tag{18}
\]

3D Spherical Coords, \( u = u(r, \theta, \phi) \):

\[
\left[ \frac{1}{r^2} \partial_r (r^2 \partial_r) + \frac{1}{r^2 \sin \theta} \partial\theta (\sin \theta \partial\theta) + \frac{1}{r^2 \sin^2 \theta} \partial\phi^2 \right] u = 0 \tag{19}
\]

\[
\partial_r^2 + \frac{2}{r} \partial_r + \frac{1}{r^2} \left( \partial\theta^2 + \frac{\cos \theta}{\sin \theta} \partial\theta^2 + \frac{1}{\sin^2 \theta} \partial\phi^2 \right) u = 0 \tag{20}
\]

\[
u(r, \theta, \phi) = \sum_i \tag{21}
\]

(w/o sources \( \Rightarrow \) scalar potential \( \nabla^2 \Phi = 0 \))

Given BCs, solns are series of harmonic functions (twice continuously differentiable)

Cartesian coords: simple harmonic functions (sin, cos, \( e^{ix} \)), hyperbolic sin cos

Cylindrical coords: for \( s \), (modified) Bessel functions; for \( \theta, z \), same as Cartesian

(Neumann, Hankel funes; Fourier-Bessel, Neumann, Kapteyn, Schlömilch series)

Spherical coords: for \( r \), associated Legendre functions; for \( \theta, \phi \), spherical harmonics

(or Legendre function of the first kind of order \( \nu \), a hypergeometric func; or Laguerre, Hermite polynomials)

2.5 Noteworthy Facts

- Poisson’s Theorem
- Maximum Principle for Harmonic Functions
2.6 Specific Problems

Laplace Eqn
Find the field due to a surface charge distribution
Find the field due to a surface potential configuration
[Examples: Jackson §5.12 Magnetic Shielding]
Jackson

Poisson Eqn

Green Functions
Find the potential of a conducting sphere in the presence of a point charge (Jackson 2.6)

3 Poisson Equation: $\nabla^2 u = f$

First of all, to what will this be relevant?

- Electrostatics: Find the potential $\Phi$ and/or the electric field $\mathbf{E}$ in a region with charge $\rho$.

- Magnetostatics:
  - Find $\mathbf{A}$ and/or $\mathbf{B}$ or $\mathbf{H}$ in the presence of a current $\mathbf{J}$. (For this example, you must use the Coulomb gauge where $\nabla \cdot \mathbf{A} = 0$, and you end up with the vector Laplacian of $\mathbf{A}$, which in Cartesian coordinates gives you a Laplacian in each component.)
  - Ferromagnetism: Find $\Phi_m$ and/or $\mathbf{B}$ or $\mathbf{H}$, given $\mathbf{M}$ and $\mathbf{J} = 0$. (Here, you’ll have a scalar Laplacian of $\Phi_m$.)

- Green Functions...

The solutions of the Poisson equation are the “homogeneous solutions” (solutions of the Laplace equation, or “complementary solutions”) plus the “inhomogeneous solutions” (the solutions that yield the “inhomogeneous term” $f$).

(w/ sources $\Rightarrow$ electric scalar potential $\nabla^2 \Phi = -\rho/\varepsilon_0$)

Given Dirichlet or Neumann BCs, Green functions will help solve eqn
(Expansions of Green functions in cylindrical and spherical coords)
General soln = complementary (homogeneous) foln (of Laplace eqn) + particular soln

4 Helmholtz Equation

$(\nabla^2 + k^2)u(x, t) = 0$

2D Cartesian coords: sin, cos
2D Polar Coords: for $s$, (modified) Bessel functions; for $\phi$, sin, cos
(hyperbolic sin cos) (Hankel functions) (use of Green functions [See Eqn World])
(spherical Bessel functions and spherical harmonics [See Wikipedia])

5 Wave Equation

$\Box^2 u = (\nabla^2 - \frac{1}{c^2} \partial^2)u(x, t) = 0, \ u \rightarrow \Phi, \mathbf{A}$
Separation of space/time variables $\Rightarrow$ Helmholtz Eqn
Planewave solns
Gauge transformations (Lorentz, Coulomb, etc.)
Green functions for wave eqn
Retarded solns for potential
Jefimenko’s generalizations of Coulomb/Biot-Savart Laws/Heaviside-Feynman Expressions for pt charge fields