Waves[*], Group Velocity[†], and Waveguides[‡]

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1 Waveguides

Waveguide problems are split into three cases based on the type of wave. The type of wave is specified as either TE (Transverse Electric),TM (Transverse Magnetic), or TEM (Transverse Electromagnetic)[1]. TE has $E_z = 0$, TM has $B_z = 0$, and TEM has both zero.

$$\text{TM: } B_z = 0 \qquad \text{TE: } E_z = 0 \qquad \text{TEM: } E_z = B_z = 0 \tag{1}$$

TEM cannot occur in a completely hollow cavity because the divergence and curl of the electric field would be zero by Gauss's Law and Faraday's Law. However, TEM modes can exist if there is a wire in the cavity, such as in a coaxial cable.

When developing the mathematics of waveguides we make two simplifying assumptions.

1. The walls of the waveguide are perfect conductors so we have the boundary conditions $E_z|_S = 0$ (since the parallel component of E is continuous and it is zero inside) and $B_\perp = 0$ or equivalently $\frac{\partial B_z}{\partial n}\big|_S = 0$ (by a subtle consequence of an equation derived from the Maxwell equations). So the boundary conditions we use are:

$$\text{TM: } E_\parallel|_S = 0 \Rightarrow E_z|_S = 0 \qquad \text{TE: } B_\perp|_S = 0 \Rightarrow \left.\frac{\partial B_z}{\partial n}\right|_S = 0 \tag{2}$$

2. The electromagnetic wave in the cavity is a monochromatic plane wave so it can be written as

$$\tilde{\mathbf{E}}(x, y, z, t) = \tilde{\mathbf{E}}_0(x, y)e^{i(kz-\omega t)}$$

$$\tilde{\mathbf{B}}(x, y, z, t) = \tilde{\mathbf{B}}_0(x, y)e^{i(kz-\omega t)}$$

[*] We need whole lecture to get into wave incidence problems.
[†] This section comes from Griffiths Quantum Page 65
[‡] This section comes from Jackson chapter 8 and Griffiths E&M section 9.5.
[1] Not to be confused with Tunneling Electron Microscope

> Note that in free space the vector $\tilde{\mathbf{E}}_0$ or $\tilde{\mathbf{B}}_0$ would not have a z component, but in a closed cavity it does for TE and TM modes.[2] This assumption is reasonable because a non-monochromatic wave can be decomposed into its monochromatic components and we are usually only concerned with plane waves rather than wave packets when it comes to waveguides.

Our goal is to find expressions for the time-dependent fields in the cavity. Naturally our starting point is with the sourceless Maxwell's equations, specifically the curl equations (which are responsible for creating the wave equation)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \to i\omega\tilde{\mathbf{B}} \qquad \nabla \times \mathbf{B} = \mu\epsilon\frac{\partial \mathbf{E}}{\partial t} \to -i\mu\epsilon\omega\tilde{\mathbf{E}}$$

If we expand these two vector equations into their component equations we get six equations that contain the components of $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$. You can see that there will be a factor of $e^{i(kz-\omega t)}$ on both sides, which we cancel. Then you can solve for $\tilde{E}_{0x}, \tilde{E}_{0y}, \tilde{B}_{0x}$, and $\tilde{B}_{0y}$ in terms of $\tilde{E}_{0z}$ and $\tilde{B}_{0z}$. This allows you to substitute back in to the set of six equations to find differential equations that only contain $\tilde{E}_{0z}(x, y)$ and $\tilde{B}_{0z}(x, y)$:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \mu\epsilon\omega^2 - k^2\right]\psi = 0$$

where

$$\psi = \begin{cases} \tilde{E}_{0z}(x, y), & \text{for TM modes;} \\ \tilde{B}_{0z}(x, y), & \text{for TE modes.} \end{cases}$$

The reason for this choice of ψ is that both $\tilde{E}_{0z}$ and $\tilde{B}_{0z}$ satisfy this equation, but one of these will be zero, so we eliminate that equation since it does not say anything new.

To make this equations look simpler, Jackson defines

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad \text{and} \qquad \gamma^2 = \mu\epsilon\omega^2 - k^2$$

Therefore the equation we want to solve is

$$(\nabla_t^2 + \gamma^2)\psi = 0 \tag{3}$$

This is just solving the 2D Helmholtz equation corresponding to a cross-sectional slice of the wave at a specific time, that is constant z and t.

The general procedure for the problem of finding the electric and magnetic fields in the cavity is

1. Solve (3) with the boundary conditions in (2).

$$(\nabla_t^2 + \gamma^2)\psi = 0 \ \text{ with } \ \text{TM: } B_z = 0 \qquad \text{TE: } E_z = 0$$

[2] Question: Why does the cavity permit non-transverse waves?

2. Get the transverse components of the same (non-transverse) field using

$$\text{TM: } \tilde{\mathbf{E}}_{0t} = \pm\frac{ik}{\gamma^2}\nabla_t\psi \quad \text{or} \quad \tilde{\mathbf{B}}_{0t} = \pm\frac{ik}{\gamma^2}\nabla_t\psi$$

3. Get the transverse components of the transverse field using

$$\text{TM: } \tilde{\mathbf{B}}_{0t} = \pm\frac{\mu\epsilon\omega}{k}\hat{z}\times\tilde{\mathbf{E}}_{0t} \quad \text{or} \quad \text{TE: } \tilde{\mathbf{E}}_{0t} = \pm\frac{\omega}{k}\hat{z}\times\tilde{\mathbf{B}}_{0t}$$

4. If you need the cutoff frequency, use the definition of γ^2 to find the value of ω below which k becomes imaginary.

$$k_c^2 = \mu\epsilon\omega^2 - \gamma_c^2 \Rightarrow \omega_c = \frac{\gamma_c}{\sqrt{\mu\epsilon}}$$

Below this frequency the solutions are evanescent waves that have exponentially decaying amplitude.

2 Group Velocity

First of all recall that the phase velocity for a monochromatic wave is simply the wavelength times the frequency

$$v_p = \lambda f = \frac{2\pi f}{2\pi/\lambda} = \frac{\omega}{k}$$

The group velocity is the velocity that the envelope of the wave travels. It is also the velocity a quantum particles travels and the velocity that energy or information travels. We will show that the group velocity for a roughly monochromatic wave with wave vector near k_0 is

$$v_{\text{group}} = \frac{d\omega}{dk}(k_0)$$

First we construct a general expression for a wave packet. Each monochromatic component has the form

$$\Psi_k(x,t) = Ae^{i(kx-\omega(k)t)}$$

where the function $\omega(k)$ is called the **dispersion relation**, which in general is the relation between the energy and momentum of a system.[3]

However, the wave packet can have an arbitrary combination of wavelengths, so we integrate over k.

$$\Psi(x,t) = \int_{-\infty}^{\infty} A(k)e^{i(kx-\omega(k)t)}dk$$

[3]This definition is from Wikipedia's article "dispersion relation".

So this is our wave packet and we want to determine how fast its envelope is traveling. The first thing we must do is it assume that $A(k)$ is narrowly peaked around $k = k_0$. The reason is that if there is a large spread in wavelengths in the packet, then it will change its shape rapidly, which prevents us from looking at a group. With this assumption, the integrand is negligible except near k_0, so we Taylor expand $w(k)$ about k_0

$$\omega(k) \cong \omega(k_0) + (k - k_0)\omega'(k_0)$$

Let $s \equiv k - k_0$ then

$$\Psi(x,t) \cong \int_{-\infty}^{\infty} A(k_0+s)e^{i[(k_0+s)x-(\omega(k_0)+s\omega'(k_0))t]}ds$$

$$= e^{i(k_0\omega'(k_0)t-\omega(k_0)t)} \int_{-\infty}^{\infty} A(k_0+s)e^{i(k_0+s)(x-\omega'(k_0)t)}ds$$

Whereas, at time $t = 0$,

$$\Psi(x,0) = \int_{-\infty}^{\infty} A(k_o+s)e^{i(k_0+s)x}ds$$

so we see that

$$\Psi(x,t) \cong e^{i(k_0\omega'(k_0)-\omega(k_0))t}\Psi(x-\omega'(k_0)t, 0)$$

Therefore

$$v_{\text{group}} = \frac{d\omega}{dk}(k_0)$$

Now what about waves that are nowhere near monochromatic? I haven't seen or thought of any good method for getting the group velocity in this case, but here is a little trick for getting an expression for the phase velocity. We are going to determine the velocity of a specific point at $x = x_0$. The assumption we make is that for an infinitesimal time interval, the portion of the wave in a small interval around x_0 is smooth and scrolls sideways without changing its shape. The validity of this assumption will not be discussed. By looking at only the point x_0, we can see $\Psi(x_0,t)$ and $\frac{\partial\Psi}{\partial x}(x_0,t)$ for all times t. With our assumptions we can conclude that the wave is effectively a straight line at some slope that is translating sideways past the point x_0. So if the line slopes upwards and we see the value of Ψ is going down, then we conclude that the line is translating to the right. But we need to calculate the velocity that it is translating at. We do this by writing the change in Ψ in two different ways:

$$d\Psi = \frac{\partial\Psi}{\partial t}dt \quad \text{and} \quad d\Psi = \frac{\partial\Psi}{\partial x}dx$$

where dx is some unknown distance that the wave traveled in the time interval dt. So we have

$$v_{\text{phase}} = \frac{dx}{dt} = \frac{\frac{d\Psi}{dt}}{\frac{d\Psi}{dx}}$$

$$= -\frac{\int_{-\infty}^{\infty} \omega(k) A(k) e^{i(kx-\omega(k)t)}\ dk}{\int_{-\infty}^{\infty} kA(k) e^{i(kx-\omega(k)t)}\ dk}$$

$$= -\frac{\int_{-\infty}^{\infty} [\int_0^k \frac{d\omega}{dk'}(k')\ dk'] A(k) e^{i(kx-\omega(k)t)}\ dk}{\int_{-\infty}^{\infty} [\int_o^k\ dk'] A(k) e^{i(kx-\omega(k)t)}\ dk} \quad \text{since} \quad \omega(0) = 0$$

$$= -\frac{\int_{-\infty}^{\infty} \int_0^k \frac{d\omega}{dk'}(k') A(k) e^{i(kx-\omega(k)t)}\ dk'\ dk}{\int_{-\infty}^{\infty} \int_o^k A(k) e^{i(kx-\omega(k)t)}\ dk'\ dk}$$

$$= -\left\langle \frac{\partial \omega}{\partial k} \right\rangle$$

for this unique definition of averaging. It is interesting that this resembles the group velocity expression.