1. *Quantum Mechanics* (Fall 2002)

A Stern-Gerlach apparatus is adjusted so that the z-component of the spin of an electron (spin-1/2) transmitted through it is $-\hbar/2$. A uniform magnetic field in the x-direction is then switched on at time $t = 0$.

(a) What are the probabilities associated with finding the different allowed values of the z-component of the spin after time T?

(b) What are the probabilities associated with finding the different allowed values of the x-component of the spin after time T?

a. $P(z\pm,t) = |\langle \psi_{z\pm} | \psi(t)\rangle|^2$

$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$ where $|\psi(0)\rangle = |\psi_{z-}\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$H = -\vec{\mu}\cdot\vec{B} = \mu_B \vec{\sigma}\cdot\vec{B} = \mu_B B\sigma_x = \mu_B B\begin{pmatrix}0&1\\1&0\end{pmatrix}$

$\hookrightarrow \mu_B = \frac{e}{2m}$ is the Bohr Magneton

We need to split up $|\psi(0)\rangle$ into eigenfunctions of H so we can replace H with the eigenvalue.

Eigenfunctions of H are eigenfunctions of $\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}$.

$|\psi(0)\rangle = \left(\sum_{i=\pm} |\psi_{xi}\rangle\langle\psi_{xi}|\right)|\psi(0)\rangle$

$= |\psi_{x+}\rangle\langle\psi_{x+}|\psi_{z-}\rangle + |\psi_{x-}\rangle\langle\psi_{x-}|\psi_{z-}\rangle$

$\sigma_x|\psi_{x+}\rangle = |\psi_{x+}\rangle \Rightarrow \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} \rightarrow \begin{pmatrix}b\\a\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix} \Rightarrow |\psi_{x+}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$\sigma_x|\psi_{x-}\rangle = -|\psi_{x-}\rangle \Rightarrow \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} \rightarrow \begin{pmatrix}b\\a\end{pmatrix} = -\begin{pmatrix}a\\b\end{pmatrix} \Rightarrow |\psi_{x-}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

So $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(1\ \ 1)\begin{pmatrix}0\\1\end{pmatrix}|\psi_{x+}\rangle + \frac{1}{\sqrt{2}}(1\ \ -1)\begin{pmatrix}0\\1\end{pmatrix}|\psi_{x-}\rangle$

$= \frac{1}{\sqrt{2}}|\psi_{x+}\rangle - \frac{1}{\sqrt{2}}|\psi_{x-}\rangle$

Therefore $|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-iHt/\hbar}|\psi_{x+}\rangle - \frac{1}{\sqrt{2}}e^{-iHt/\hbar}|\psi_{x-}\rangle$

$= \frac{1}{\sqrt{2}}e^{-i\mu_B Bt/\hbar}|\psi_{x+}\rangle - \frac{1}{\sqrt{2}}e^{i\mu_B Bt/\hbar}|\psi_{x-}\rangle$

Now we need to apply $\langle\psi_{z+}|$,

$\langle\psi_{z+}|\psi_{x+}\rangle = (1\ \ 0)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}$

$\langle\psi_{z+}|\psi_{x-}\rangle = (1\ \ 0)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}$

$P(z+,T) = |\langle\psi_{z+}|\psi(T)\rangle|^2 = \left|\frac{1}{2}e^{-i\mu_B BT/\hbar} - \frac{1}{2}e^{i\mu_B BT/\hbar}\right|^2$

$= |-\sin(\mu_B BT/\hbar)|^2 = \sin^2(\mu_B BT/\hbar)$

$\Rightarrow P(z-,T) = \cos^2(\mu_B BT/\hbar)$ since $\sin^2(\theta)+\cos^2(\theta) = 1$.

b. $P(x+,T) = |\langle\psi_{x+}|\psi(T)\rangle|^2 = \left|\frac{1}{\sqrt{2}}e^{-i\mu_B BT/\hbar}\right|^2 = \frac{1}{2}$

$P(x-,T) = |\langle\psi_{x-}|\psi(T)\rangle|^2 = \left|\frac{-1}{\sqrt{2}}e^{i\mu_B BT/\hbar}\right|^2 = \frac{1}{2}$