5. *Quantum Mechanics* (Fall 2003)

A sample of hydrogen atoms in the ground state is placed between the plates of a parallel plate capacitor. A voltage pulse is applied to the capacitor at $t = 0$ to produce a homogeneous electric field, ε, between the plates of:

$$\varepsilon = 0, \qquad (t < 0)$$
$$\varepsilon = \varepsilon_0 \exp(-t/\tau), \quad (t > 0),$$

where τ is a constant. A long time compared to τ passes.

(a) To first order, calculate the fraction of atoms in the $2p$ ($m = 0$) state.

(b) To first order, what is the fraction of atoms in the $2s$ state?

You may find the following helpful. The normalized radial wavefunctions of the hydrogen atom are:

$$R_{10}(r) = 2\left(\frac{Z}{a}\right)^{3/2} \exp\left(-\frac{Zr}{a}\right)$$
$$R_{20}(r) = \frac{1}{\sqrt{2}}\left(\frac{Z}{a}\right)^{3/2}\left(1 - \frac{Zr}{2a}\right) \exp\left(-\frac{Zr}{2a}\right)$$
$$R_{21}(r) = \frac{1}{2\sqrt{6}}\left(\frac{Z}{a}\right)^{5/2} r \exp\left(-\frac{Zr}{2a}\right)$$

where r is the radial coordinate, a is the Bohr radius, and $Z = 1$ for a hydrogen atom. The first spherical harmonics are:

$$Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}} \qquad Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos(\theta) \qquad Y_{1\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin(\theta)\exp(\pm i\phi)$$

A useful integral may be:

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$

a. $P(f) = |\langle \Psi_f | \Psi \rangle|^2$ where $\langle \Psi_f | \Psi \rangle = \delta_{fi} - \frac{i}{\hbar}\int_0^t \langle \Phi_f | H'(t') | \Phi_i \rangle e^{i\omega_{fi}t'} dt'$

and $H'(t) = e\Phi = -eEz = -eE_0 e^{-t/\tau} z$ so

$$\langle 210 | H'(t') | 100 \rangle = -eE_0 e^{-t/\tau} \langle 210 | z | 100 \rangle$$

$$\langle 210 | z | 100 \rangle = \int_0^\infty \int_0^\pi \int_0^{2\pi} R_{21}(r) Y_{10}(\theta,\phi)\, r\cos(\theta)\, R_{10}(r) Y_{00}(\theta,\phi)\, r^2 \sin(\theta)\, dr\, d\theta\, d\phi$$

$$= \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{1}{2\sqrt{6}} a^{-5/2} r e^{-r/2a} \sqrt{\frac{3}{4\pi}} \cos(\theta)\, r\cos(\theta)\, 2a^{-3/2} e^{-r/a} \frac{1}{\sqrt{4\pi}} r^2 \sin(\theta)\, dr\, d\theta\, d\phi$$

$$= \frac{2\pi}{4\pi} \frac{1}{\sqrt{2}} a^{-4} \int_0^\pi \cos^2(\theta)\sin(\theta)\, d\theta \int_0^\infty r^4 e^{-3r/2a} dr$$

$$= \frac{a^{-4}}{2\sqrt{2}} \left(\frac{2}{3}\right) 4! = \frac{2^8}{\sqrt{2}\, 3^5} a$$

Therefore $P(|210\rangle) = \left| -\frac{i}{\hbar}\left(-eE_0 \frac{2^8}{\sqrt{2}\,3^5} a\right) \int_0^t e^{-t'/\tau} e^{i\omega_{fi}t'} dt' \right|^2$

$$= \frac{2^{15}}{3^{10}} \frac{a^2 e^2 E_0^2}{\hbar^2} \left| \int_0^t e^{-t'/\tau + i\omega_{fi}t'} dt' \right|^2 = \frac{2^{15}}{3^{10}} \frac{a^2 e^2 E_0^2}{\hbar^2} \left| \frac{e^{-t/\tau} e^{i\omega_{fi}t} - e^0}{-\frac{1}{\tau} + i\omega_{fi}} \right|^2$$

$$= \frac{2^{15}}{3^{10}} \frac{a^2 e^2 E_0^2}{\hbar^2} \frac{1}{\left(\frac{1}{\tau}\right)^2 + \omega_{fi}^2} \quad \text{where } \omega_{fi} = \frac{E_2 - E_1}{\hbar} = -\frac{3E_1}{4\hbar}$$

b. $P(|200\rangle) = 0$ to first order by the m-selection rule.