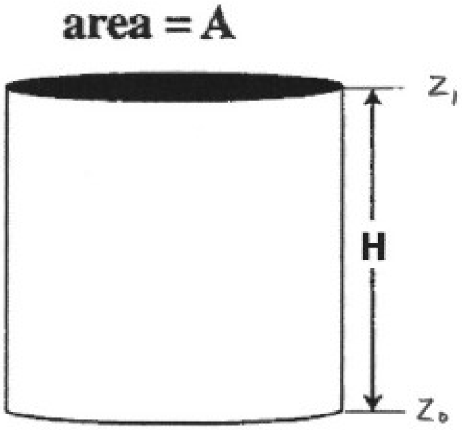
7. *Statistical Mechanics and Thermodynamics* (Fall 2003)

A gas of noninteracting particles fills a cylindrical container that has cross-sectional area A and height H. Each particle has mass m, and is subject to the gravitational field at the surface of the Earth. The circular bottom and top of the container are parallel to the surface of the Earth. There are N particles in the container, and the temperature of the container is T.



(a) Find the partition function of the gas.

(b) What is the pressure of the gas at the top of the container?

(c) What is the pressure of the gas at the bottom of the container?

(d) Finally, what is the difference between the pressure at the bottom of the container and the pressure at the top of the container? Interpret the answer that you get.

a. $\mathcal{Z} = \sum_r e^{-\beta E_r} \cong \frac{1}{h^3}\iint e^{-(\frac{p^2}{2m}+mgz)\beta}\, d^3p\, d^3x$

$= \frac{1}{h^3} A \iint e^{-(p^2/2m+mgz)\beta}\, 4\pi p^2 dp\, dz$

$= \frac{1}{h^3} 4\pi A \left(\frac{2m}{\beta}\right)^{3/2} \int_0^\infty u^2 e^{-u^2} du \int_{z_0}^{z_1} e^{-mg\beta z} dz$

$= \frac{1}{h^3} 4\pi A \left(\frac{2m}{\beta}\right)^{3/2} \frac{\sqrt{\pi}}{4} \left(-\frac{1}{mg\beta}\right)\left(e^{-mg\beta z_1} - e^{-mg\beta z_0}\right)$

$= \frac{A}{mg\beta}\left(\frac{2m\pi}{\beta h^2}\right)^{3/2}\left(e^{-mg\beta z_0} - e^{-mg\beta z_1}\right)$

$\Rightarrow \ln(Z) = \ln\left(\frac{\mathcal{Z}^N}{N!}\right) = N\ln\left(\frac{A}{mg\beta}\right) + \frac{3}{2}N\ln\left(\frac{2m\pi}{\beta h^2}\right) + N\ln\left(e^{-mg\beta z_0} - e^{-mg\beta z_1}\right) - N\ln(N)$

b. $P_{top} = \frac{1}{\beta}\frac{\partial \ln(Z)}{\partial V_{top}} = \frac{1}{\beta}\frac{\partial \ln(Z)}{\partial z_1}\frac{\partial z_1}{\partial V} = \frac{1}{\beta A}\frac{\partial \ln(Z)}{\partial z_1} = \frac{N}{\beta A}\frac{mg\beta e^{-mg\beta z_1}}{e^{-mg\beta z_0} - e^{-mg\beta z_1}}$

$= \frac{mgN}{A}\frac{1}{e^{mg\beta(z_1-z_0)}-1} = \frac{mgN}{A}\frac{1}{e^{mg\beta H}-1}$

c. $P_{bot} = \frac{1}{\beta}\frac{\partial \ln(Z)}{\partial V_{bot}} = \frac{1}{\beta}\frac{\partial \ln(Z)}{\partial z_0}\frac{\partial z_0}{\partial V} = -\frac{1}{\beta A}\frac{\partial \ln(Z)}{\partial z_0} = \frac{-N}{\beta A}\frac{-mg\beta e^{-mg\beta z_0}}{e^{-mg\beta z_0} - e^{-mg\beta z_1}}$

$= \frac{mgN}{A}\frac{1}{1-e^{mg\beta(z_0-z_1)}} = \frac{mgN}{A}\frac{1}{1-e^{-mg\beta H}}$

d. $\Delta p = p_{bot} - p_{top} = \frac{mgN}{A}\left(\frac{1}{1-e^{-mg\beta H}} - \frac{1}{e^{mg\beta H}-1}\right)$

$= \frac{mgN}{A}\left(\frac{e^{mg\beta H}}{e^{mg\beta H}-1} - \frac{1}{e^{mg\beta H}-1}\right) = \frac{mgN}{A} = \frac{mgN}{V}H = nmgH$

The increased pressure at the bottom of the container is due to the force from the weight of the gas above it.