8. *Electricity and Magnetism* (Fall 2003)

Consider a vacuum diode which is a parallel plate capacitor (in vacuum) with plate area A and plate separation d. The cathode plate, which is at $\phi = 0$, is heated as to thermionically emit electrons which then travel to the anode plate (at $\phi = V$) (this arrangement acts as a diode due to the fact that in reverse bias, no charges will flow). Assume a steady-state bias V and diode current I. You may model the electrons in the diode as a cold fluid with density $n(x)$ and velocity $v(x)$. You may assume that the electrons are born from the cathode with zero velocity.

(a) Find the 1-D potential distribution in the diode, $\phi(x)$. (Hint: Try a power law solution.)

(b) Find the diode current as a function of bias voltage V.

(c) What unphysical result is caused by the assumption that electrons are born from the cathode with zero velocity?

a. We use $I = \int \vec{J} \cdot d\vec{a}$, $\vec{J}(x) = \rho(x)\vec{v}(x)$, and $\rho(x) = en(x)$ to write $I(x) = \int \vec{J}(x) \cdot d\vec{a} = AJ(x) = A\rho(x)v(x) = Aen(x)v(x)$

By conservation of energy, $\frac{1}{2}mv^2(x) = e\Phi(x)$ where $\Phi(0) = 0$, $\Phi(d) = V$

so by Gauss' Law $\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \Rightarrow$

$$\frac{\partial^2\Phi}{\partial x^2} = -\frac{I}{\epsilon_0 A v(x)} = \frac{1}{\epsilon_0}\frac{I}{A}\sqrt{\frac{m}{2e\Phi(x)}}$$

Now assume a power law solution: $\Phi(x) = Cx^n$

$$\Phi(d) = V \Rightarrow V = Cd^n \Rightarrow C = Vd^{-n} \Rightarrow \Phi(x) = V\left(\frac{x}{d}\right)^n$$

Now we find n by matching exponents of x:

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(\frac{V}{d^n}x^n\right) = n\frac{V}{d^n}\frac{\partial}{\partial x}(x^{n-1}) = n(n-1)\frac{V}{d^n}x^{n-2}$$

$$\frac{\partial^2\Phi}{\partial x^2} = -\frac{1}{\epsilon_0}\frac{I}{A}\sqrt{\frac{m}{2e\Phi(x)}} = -\frac{1}{\epsilon_0}\frac{I}{A}\sqrt{\frac{m}{2e}}V^{-1/2}d^{n/2}x^{-n/2}$$

$$\Rightarrow n-2 = -n/2 \Rightarrow \frac{3}{2}n = 2 \Rightarrow n = \frac{4}{3}$$

Therefore $\Phi(x) = V\left(\frac{x}{d}\right)^{4/3}$

b. $$\frac{\partial^2\phi}{\partial x^2} = -\frac{1}{\epsilon_0}\frac{I}{A}\sqrt{\frac{m}{2e\Phi(x)}} = n(n-1)\frac{V}{d^n}x^{n-2} = \frac{4}{9}\frac{V}{d^{4/3}}x^{-2/3}$$

At $x = d$, $-\frac{1}{\epsilon_0}\frac{I}{A}\sqrt{\frac{m}{2eV}} = \frac{4}{9}Vd^{-2}$

$$\Rightarrow I = -\frac{4}{9}\epsilon_0 V\frac{A}{d^2}\sqrt{\frac{2eV}{m}} = -\frac{4}{9}\epsilon_0 V^{3/2}\frac{A}{d^2}\sqrt{\frac{2e}{m}}$$

c. $I = Aen(x)v(x)$ must be independent of x, so it is nonzero everywhere if the current is nonzero, so if $v(x) = 0$, then $n(x) = \infty$, which is unphysical.