13. *Statistical Mechanics and Thermodynamics* (Fall 2004)

Consider the Landau-Ginzburg free energy functional for a magnet with magnetization M:

$$F(M) = \frac{1}{2}rM^2 + uM^4 - hM$$

M takes values $M \in [-\infty, \infty]$. (The rotational symmetry of the magnet is broken by the crystal so that M is a scalar, not a vector.) $r = a(T - T_c)$, u is only weakly dependent on T, and h is the magnetic field. We will make the mean-field approximation that M is equal to the value which minimizes $F(M)$, and $F(M)$ is given by its minimum value.

(a) For $T > T_c$ and $h = 0$, what value of M minimizes F? For $T < T_c$ and $h = 0$, what value of M minimizes F?

(b) For $h = 0$, the specific heat takes the asymptotic form $C \sim |T - T_c|^{-\alpha}$ as $T \to T_c$. What is α?

(c) At $T = T_c$, $M \sim h^\delta$. What is δ?

a. Mean-field approximation $\Rightarrow \frac{\partial F}{\partial M} = rM + 4uM^3 - h = 0$

And $h = 0 \Rightarrow 4uM^3 = -rM \Rightarrow M = 0$ or $M = \sqrt{-\frac{r}{4u}}$

If $T > T_c$, then $r = a(T - T_c) > 0$, so all terms in F are positive, so $M = 0$ minimizes F.

If $T < T_c$, then $r = a(T - T_c) < 0$, so $M = \sqrt{\frac{-r}{4u}}$ is a real solution that makes $F(M) = \frac{1}{2}r\left(-\frac{r}{4u}\right) + u\left(\frac{r^2}{16u^2}\right) = -\frac{r^2}{16u}$ which is less than zero, so $M = \sqrt{-\frac{r}{4u}}$ minimizes F.

b. The free energy functional is a type of Gibbs free energy so we use $\left(\frac{\partial G}{\partial T}\right)_p = -S$ and $C_v = T\left(\frac{\partial S}{\partial T}\right)_v$

We assume that $T \to T_c$ from below T_c because the Mean field approximation used here gives a trivial result otherwise

$$G = F(M) = \frac{1}{2}a(T - T_c)\frac{-a(T-T_c)}{4u} + u\frac{a^2(T-T_c)^2}{16u^2} = -\frac{a^2(T-T_c)^2}{16u^2}$$

$$\Rightarrow C_v = T\left(\frac{\partial S}{\partial T}\right)_v = T\frac{\partial^2 G}{\partial T^2} = T\frac{\partial}{\partial T}\left(-\frac{a^2(T-T_c)}{8u^2}\right) = -\frac{a^2}{8u^2}T$$

$$= -\frac{a^2}{8u^2}\left[(T-T_c) + T_c\right] \cong -\frac{a^2}{8u^2}T_c \quad \text{since } |T - T_c| << T_c$$

in the asymptotic limit, so $C_v \sim |T - T_c|^0 \Rightarrow \alpha = 0$

c. $F(M)\big|_{T=T_c} = uM^4 - hM$

$$\frac{\partial F}{\partial M}\bigg|_{T=T_c} = 4uM^3 - h = 0 \Rightarrow h = 4uM^3 \Rightarrow M = \left(\frac{h}{4u}\right)^{1/3} \Rightarrow \delta = \frac{1}{3}$$