14. *Statistical Mechanics and Thermodynamics* (Fall 2004)

Consider black body radiation at temperature $T$. What is the average energy per photon in units of $kT$?

You may find the following formulae useful:

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} \approx 6.5; \qquad \int_0^\infty \frac{x^2 dx}{e^x - 1} \approx 2.4$$

$$\epsilon = pc = \hbar k c = \hbar c \sqrt{\left(\frac{n_x \pi}{L}\right)^2 + \left(\frac{n_y \pi}{L}\right)^2 + \left(\frac{n_z \pi}{L}\right)^2}$$

$$= \frac{\hbar c \pi}{L} \sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{\hbar c \pi}{L} n$$

$$\Rightarrow n = \frac{L}{\hbar c \pi} \epsilon \quad \text{and} \quad dn = \frac{L}{\hbar c \pi} d\epsilon$$

$$\rho(\epsilon) d\epsilon = \frac{1}{8} 4\pi n^2 dn = \frac{1}{2}\pi \left(\frac{L}{\hbar c \pi}\right)^3 \epsilon^2 d\epsilon = \frac{V}{2\pi^2} \frac{\epsilon^2}{(\hbar c)^3} d\epsilon$$

$$\langle \epsilon \rangle = \frac{\int_0^\infty \epsilon f(\epsilon) \rho(\epsilon) d\epsilon}{\int_0^\infty f(\epsilon) \rho(\epsilon) d\epsilon} \quad \text{where } f(\epsilon) = \frac{1}{e^{\beta\epsilon} - 1}$$

$$\int_0^\infty \epsilon f(\epsilon) \rho(\epsilon) d\epsilon = \frac{V}{2\pi^2} \frac{1}{(\hbar c)^3} \int_0^\infty \frac{\epsilon^3}{e^{\beta\epsilon} - 1} d\epsilon$$

Let $x = \beta\epsilon \Rightarrow d\epsilon = \frac{1}{\beta} dx$

$$= \frac{V}{2\pi^2} \frac{1}{(\hbar c)^3} \frac{1}{\beta^4} \int_0^\infty \frac{x^3}{e^x - 1} dx$$

$$\cong \frac{V}{2\pi^2} \frac{1}{(\hbar c)^3} (kT)^4 (6.5)$$

$$\int_0^\infty f(\epsilon) \rho(\epsilon) d\epsilon = \frac{V}{2\pi^2} \frac{1}{(\hbar c)^3} \int_0^\infty \frac{\epsilon^2}{e^{\beta\epsilon} - 1} d\epsilon$$

Let $x = \beta\epsilon \Rightarrow d\epsilon = \frac{1}{\beta} dx$

$$= \frac{V}{2\pi^2} \frac{1}{(\hbar c)^3} \frac{1}{\beta^3} \int_0^\infty \frac{x^2}{e^x - 1} dx$$

$$\cong \frac{V}{2\pi^2} \frac{1}{(\hbar c)^3} (kT)^3 (2.4)$$

$$\Rightarrow \langle \epsilon \rangle \cong \frac{6.5}{2.4} (kT) \cong 2.7 kT$$