3. *Quantum Mechanics* (Fall 2004)

A positron has the same mass m as the electron, but the opposite charge. Consider a set of states containing one electron and one positron. A complete set of these states can be labeled $|\mathbf{r}_+, \mathbf{r}_-\rangle$, where $\mathbf{r}_+$ and $\mathbf{r}_-$ are the positions of the positron and electron, respectively. Normalize these states so that

$$\langle \mathbf{r}_+, \mathbf{r}_- | \mathbf{r}'_+, \mathbf{r}'_- \rangle = \delta_3(\mathbf{r}'_+ - \mathbf{r}_+)\, \delta_3(\mathbf{r}'_- - \mathbf{r}_-)$$

Then if the system is in any state $|\psi\rangle$, the wave function is

$$\psi(\mathbf{r}_+, \mathbf{r}_-) = \langle \mathbf{r}_+, \mathbf{r}_- | \psi \rangle$$

In this problem ignore spin.

(a) In terms of $\psi(\mathbf{r}_+, \mathbf{r}_-)$, what is the probability that at least one of the two particles is farther than a distance b from the origin?

(b) Write down the Hamiltonian for this electron-positron system, including the electrostatic (Coulomb) interactions between the two particles.

(c) Let $\mathbf{r} = \mathbf{r}_+ - \mathbf{r}_-$ and $\mathbf{R} = \frac{1}{2}(\mathbf{r}_+ + \mathbf{r}_-)$. Write the Hamiltonian in terms of the new coordinates and their canonically conjugate momenta $\mathbf{p}$ and $\mathbf{P}$.

(d) The bound electron-positron system is called *positronium*. For states with zero total momentum, write a formula for the possible negative values of the energy[1]. What is the approximate numerical value, in electron volts, of the ground state energy?

(e) Define the *charge conjugation* operator C on this system by

$$C\, |\mathbf{r}_+, \mathbf{r}_-\rangle = |\mathbf{r}_-, \mathbf{r}_+\rangle$$

Show that C commutes with the Hamiltonian. What is the eigenvalue of C on the state of lowest energy?

a. $P = 1 - \int_0^{2\pi}\int_0^{\pi}\int_0^{b}\left[\int_0^{2\pi}\int_0^{\pi}\int_0^{b} |\psi(\vec r_+, \vec r_-)|^2 r_+^2 \sin(\theta_+)\, dr_+ d\theta_+ d\phi_+\right] r_-^2 \sin(\theta_-)\, dr_- d\theta_- d\phi_-$

b. $H = \frac{p_+^2}{2m} + \frac{p_-^2}{2m} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{|\vec r_+ - \vec r_-|}$

c. $T = \frac{p_+^2}{2m} + \frac{p_-^2}{2m} = \frac{(m\dot{\vec r}_+)^2}{2m} + \frac{(m\dot{\vec r}_-)^2}{2m} = \frac{1}{2}m\dot{\vec r}_+^{\,2} + \frac{1}{2}m\dot{\vec r}_-^{\,2} = \frac{1}{2}m\left(2\dot{\vec R}^2 + \frac{1}{2}\dot{\vec r}^2\right)$

$V = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{|\vec r_+ - \vec r_-|} = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{|\vec r|}$

$\vec p = \frac{\partial L}{\partial \dot{\vec r}} = \frac{\partial}{\partial \dot{\vec r}}(T - V) = \frac{1}{2}m\dot{\vec r} \qquad \vec P = \frac{\partial L}{\partial \dot{\vec R}} = 2m\dot{\vec R}$

$H = T + V = m\left(\frac{\vec P}{2m}\right)^2 + \frac{1}{4}m\left(\frac{2\vec p}{m}\right)^2 - \frac{1}{4\pi\epsilon_0}\frac{e^2}{|\vec r|} = \frac{P^2}{4m} + \frac{p^2}{m} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{|\vec r|}$

d. Total Momentum zero $\Rightarrow P = 0 \Rightarrow H = \frac{p^2}{m} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{|\vec r|}$

Which is Hydrogen atom with $m \to \frac{m}{2}$

$\Rightarrow E_1 = -\frac{mk^2e^4}{4n^2\hbar^2}\Big|_{n=1} = -\frac{mk^2e^4}{4\hbar^2} = \frac{1}{2}(-13.6\,\text{eV}) = -6.8\,\text{eV}$

e. $C\vec r|\vec r_+', \vec r_-'\rangle = C(\vec r_+' - \vec r_-')|\vec r_+', \vec r_-'\rangle = (\vec r_+' - \vec r_-')|\vec r_-', \vec r_+'\rangle$

$\vec r C|\vec r_+', \vec r_-'\rangle = \vec r|\vec r_-', \vec r_+'\rangle = (\vec r_-' - \vec r_+')|\vec r_-', \vec r_+'\rangle \Rightarrow C\vec r = -\vec r C$

True for all states since its true for an entire basis.

Similary $C\vec R = \vec R C$, so $C\vec p = C\left(\frac{2\dot{\vec r}}{m}\right) = -\left(\frac{2\dot{\vec r}}{m}\right)C = -\vec p C$, $C\vec P = C(2m\dot{\vec R}) = (2m\dot{\vec R})C = \vec P C$

So $CH = \left[\frac{P^2}{4m} + \frac{(-p)^2}{m} - \frac{1}{4\pi\epsilon_0}\frac{q^2}{|-\vec r|}\right]C = HC \Rightarrow [H, C] = 0 \Rightarrow$ C eigenstates are H eigenstates

Like Hydrogen, lowest state is spherically symmetric $\Rightarrow C|\psi(\vec r)\rangle = |\psi(-\vec r)\rangle = |\psi(\vec r)\rangle \Rightarrow \lambda = 1$