11. *Electricity and Magnetism* (Spring 2003)

Secret Circuit: A two-terminal "black box" is given to you. Inside the box a circuit is attached to the terminals which is known to contain a lossless inductor L, a lossless capacitor C, and a resistor R. When a 1.5 Volt battery is connected across the terminals, a current of 1.5 milliamperes flows. When an AC voltage of 1.0 Volt (RMS) at a frequency of 60 Hz is connected, a current of 0.01 amperes (RMS) flows. As the AC frequency is increased while the applied voltage is maintained constant, the current is found to go through a maximum exceeding 100 amperes at $\nu = 1000$ Hz. What is the circuit inside the box? What are the values of R, L, and C?

The requirements on the circuit are

- Resistor of resistance $R = \frac{V}{I} = \frac{1.5V}{1.5mA} = 1000\Omega$ is in it
- Inductor L is not in parallel or it would short circuit
- Capacitor C is not in series or it would block D.C.
- Inductor and Capacitor are in series so there is a resonant frequency at which the impedance goes to zero

So the possible circuits are

Now we solve for C and L by applying the conditions

$$Z(\omega = 2\pi \cdot 60\,Hz) = \frac{V}{I} = \frac{1.0V}{0.01A} = 100\Omega$$

$$Z(\omega = 2\pi \cdot 1000\,Hz) = \frac{V}{I} = \frac{1.0V}{100A} \cong 0$$

For the first circuit, $\frac{1}{Z} = \frac{1}{R} + \frac{1}{Z_L + Z_C} = \frac{1}{R} + \frac{1}{i\omega L + \frac{1}{i\omega C}} = \frac{1}{R} + \frac{i\omega C}{1 - \omega^2 LC}$

So there is a frequency of zero impedance at $\omega^2 = \frac{1}{LC} \Rightarrow \omega = \sqrt{\frac{1}{LC}}$

$$\Rightarrow LC = \frac{1}{\omega^2} = \frac{1}{4\pi^2 \cdot 10^6\,Hz^2}$$

Now using the first condition, $\frac{1}{100\Omega} = \left| \frac{1}{1000\Omega} + \frac{i(2\pi \cdot 60Hz)C}{1 - (2\pi \cdot 60Hz)^2 LC} \right|$

$$\Rightarrow \frac{1}{100\Omega} \cong \left| \frac{1}{1000\Omega} + i(2\pi \cdot 60Hz)C \right|$$

$$\Rightarrow 10^{-4}\Omega^{-2} \cong 10^{-6}\Omega^{-2} + (2\pi \cdot 60Hz)^2 C^2.$$

$$\Rightarrow 10^{-4}\Omega^{-2} \cong (2\pi \cdot 60Hz)^2 C^2$$

$$\Rightarrow 10^{-2}\Omega^{-1} \cong 2\pi \cdot 60\,Hz\,C$$

$$\Rightarrow C \cong \frac{1}{12\pi}\,mF$$

$$\Rightarrow L = \frac{1}{4\pi^2 \cdot 10^6 Hz^2} \frac{1}{C} = \frac{12\pi \cdot 10^3}{4\pi^2 \cdot 10^6} H = \frac{3}{\pi}\,mH$$