12. *Electricity and Magnetism* (Spring 2003)

(a) Show that the field inside a sphere of uniformly magnetized material ($\mathbf{M} = M\,\hat{\mathbf{z}}$) is:

$$\mathbf{B} = \frac{2}{3}\mu_0 M \hat{\mathbf{z}}$$

(b) A sphere of material with linear magnetic susceptibility χ_m is placed in a region of uniform magnetic field $B_o\hat{\mathbf{z}}$. Using the above result, find the magnetic field inside the sphere.

a. $\vec{\nabla}\times\vec{H} = \rho_f = 0 \Rightarrow \vec{H}$ is curl free $\Rightarrow \vec{H} = -\vec{\nabla}\Phi_M$ for some scalar field Φ_M

$\vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{\nabla}\Phi_M = -\nabla^2\Phi_M$ and $\vec{\nabla}\cdot\vec{H} = \vec{\nabla}(\frac{1}{\mu_0}\vec{B}-\vec{M}) = -\vec{\nabla}\cdot\vec{M} \Rightarrow \nabla^2\Phi_M = \vec{\nabla}\cdot\vec{M}$

Since the magnetization is uniform, only the boundary contributes

$$\Phi_M = \frac{1}{4\pi}\int_S \frac{K_b(\vec{x}')}{|\vec{x}-\vec{x}'|}da' = \frac{1}{4\pi}\int_S \frac{\vec{M}(\vec{x}')\cdot\hat{n}}{|\vec{x}-\vec{x}'|}da'$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{M\cos(\theta')}{|\vec{x}-\vec{x}'|} a^2 \sin(\theta')\,d\theta' d\phi'$$

$$= \frac{Ma^2}{4\pi}\int \frac{\cos(\theta')}{|\vec{x}-\vec{x}'|}d\Omega'$$

Now we use the expansion $\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$

$$\Phi_M = \frac{Ma^2}{4\pi}\int 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)\cos(\theta')\,d\Omega'$$

$$= Ma^2\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\phi)\int Y_{lm}^*(\theta',\phi')\left(\sqrt{\tfrac{4\pi}{3}}Y_{10}(\theta',\phi')\right)d\Omega'$$

since $Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos(\theta)$. Now $\int Y_{l'm'}^*(\theta,\phi)Y_{lm}(\theta,\phi)\,d\Omega = \delta_{l'l}\delta_{m'm}$,

$$\Phi_M = Ma^2\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\phi)\sqrt{\tfrac{4\pi}{3}}\,\delta_{l1}\delta_{m0}$$

$$= Ma^2\left(\frac{1}{3}\frac{r_<}{r_>^2}\right)Y_{10}(\theta,\phi)\sqrt{\tfrac{4\pi}{3}} = \frac{1}{3}Ma^2\frac{r_<}{r_>^2}\cos(\theta)$$

And $r_>, r_<$ are the greater and lesser between r and a, so inside $r<a$:

$$\Phi_M = \frac{1}{3}Ma^2\frac{r}{a^2}\cos(\theta) = \frac{1}{3}Mr\cos(\theta) = \frac{1}{3}Mz$$

Therefore $\vec{H} = -\vec{\nabla}\Phi_M = -\frac{1}{3}M\hat{z} = -\frac{1}{3}\vec{M}$

and $\vec{H} = \frac{1}{\mu_0}\vec{B}-\vec{M} \Rightarrow \vec{B} = \mu_0(\vec{H}+\vec{M}) = \frac{2}{3}\mu_0\vec{M} = \frac{2}{3}\mu_0 M\hat{z}$

b. Note that $\vec{B} = \vec{B}_0 + \vec{B}_{sphere}$

$$\vec{M} = \chi_m\vec{H} = \chi_m\left(\frac{1}{\mu_0}\vec{B}-\vec{M}\right) = \chi_m\left(\frac{1}{\mu_0}(\vec{B}_0+\vec{B}_{sphere})-\vec{M}\right)$$

$$= \chi_m\left(\frac{1}{\mu_0}\vec{B}_0 + \frac{2}{3}\vec{M} - \vec{M}\right) = \chi_m\left(\frac{1}{\mu_0}\vec{B}_0 - \frac{1}{3}\vec{M}\right)$$

$$\Rightarrow \left(1+\frac{\chi_m}{3}\right)\vec{M} = \frac{\chi_m}{\mu_0}\vec{B}_0 \Rightarrow \vec{M} = \frac{\chi_m}{\mu_0}\vec{B}_0\left(1+\frac{\chi_m}{3}\right)^{-1}$$

Therefore $\vec{B} = \vec{B}_0 + \vec{B}_{sphere} = \vec{B}_0 + \frac{2}{3}\mu_0\vec{M} = \vec{B}_0 + \frac{2}{3}\frac{\chi_m}{1+\frac{\chi_m}{3}}\vec{B}_0$

$$= \frac{1+\frac{\chi_m}{3}}{1+\frac{\chi_m}{3}}\vec{B}_0 + \frac{\frac{2\chi_m}{3}}{1+\frac{\chi_m}{3}}\vec{B}_0 = \frac{1+\chi_m}{1+\frac{\chi_m}{3}}\vec{B}_0$$