13. *Statistical Mechanics and Thermodynamics* (Spring 2003)

Consider a d-dimensional material in which the important excitation are non-conserved bosons, and assume that the dispersion relation for these bosons is $\omega = ak^3$, where k is the wave vector's amplitude and a is a constant. The low temperature specific heat goes as T^q. What is the value of the power, q? Note: The dimensionality, d, of the material is not necessarily equal to three.

$C_v = \left(\frac{dE}{dT}\right)_v$ so we calculate $E = \int_0^\infty \epsilon f(\epsilon)\rho(\epsilon)\,d\epsilon$

where $f(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}-1} = \frac{1}{e^{\beta\epsilon}-1}$ since $\mu=0$ for non-conserved particles

Now we find the energy density of states.

$$\rho(\epsilon)d\epsilon = \rho(\vec{n})\,d^d n \propto n^{d-1}dn$$

$$\epsilon \propto \omega = ak^3 = a\left(\frac{n\pi}{L}\right)^3 \Rightarrow n^3 = \frac{L^3}{a\pi^3}\epsilon \Rightarrow n = \frac{L}{\pi}\left(\frac{\epsilon}{a}\right)^{1/3}$$

$$\Rightarrow dn = \frac{1}{3}\frac{L}{\pi}a^{-1/3}\epsilon^{-2/3}d\epsilon$$

$$\Rightarrow \rho(\epsilon)d\epsilon \propto \left(\frac{L}{\pi}\left(\frac{\epsilon}{a}\right)^{1/3}\right)^{d-1}\left(\frac{1}{3}\frac{L}{\pi}a^{-1/3}\epsilon^{-2/3}d\epsilon\right) \propto \epsilon^{d/3-1}d\epsilon$$

Therefore $E \propto \int_0^\infty \epsilon \frac{1}{e^{\beta\epsilon}-1}\epsilon^{d/3-1}\,d\epsilon$

$$= \int_0^\infty \frac{\epsilon^{d/3}}{e^{\beta\epsilon}-1}\,d\epsilon$$

Let $x = \beta\epsilon$, $dx = \beta\,d\epsilon$

$$\Rightarrow E \propto \left(\frac{1}{\beta}\right)^{d/3+1}\int_0^\infty \frac{x^{d/3}}{e^x-1}\,dx$$

$$\Rightarrow E \propto T^{d/3+1}$$

Therefore $C_v = \left(\frac{dE}{dT}\right)_v \propto T^{d/3}$