2. *Quantum Mechanics* (Spring 2003)

Consider two $s = 1/2$ spins interacting through the Hamiltonian

$$H = J\sigma_1^z\sigma_2^z + h(\sigma_1^x + \sigma_2^x)$$

What is the ground state energy?

We choose to work in the basis $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$

so we express $|\psi\rangle$ as the 4-component spinor $|\psi\rangle = \begin{pmatrix} \langle ++|\psi\rangle \\ \langle +-|\psi\rangle \\ \langle -+|\psi\rangle \\ \langle --|\psi\rangle \end{pmatrix}$

No we consider the action of the terms of the Hamiltonian on the base kets. Note $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$$\sigma_1^z\sigma_2^z|++\rangle = |++\rangle \qquad (\sigma_1^x+\sigma_2^x)|++\rangle = \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 + \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 = |-+\rangle + |+-\rangle$$

$$\sigma_1^z\sigma_2^z|+-\rangle = -|+-\rangle \qquad (\sigma_1^x+\sigma_2^x)|+-\rangle = \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 = |--\rangle + |++\rangle$$

$$\sigma_1^z\sigma_2^z|-+\rangle = -|-+\rangle \qquad (\sigma_1^x+\sigma_2^x)|-+\rangle = \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 + \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 = |++\rangle + |--\rangle$$

$$\sigma_1^z\sigma_2^z|--\rangle = |--\rangle \qquad (\sigma_1^x+\sigma_2^x)|--\rangle = \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 = |+-\rangle + |-+\rangle$$

$$H \doteq J\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + h\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} J & h & h & 0 \\ h & -J & 0 & h \\ h & 0 & -J & h \\ 0 & h & h & J \end{pmatrix}$$

Now we find the energy eigenvalues by solving $\det(H-\lambda I) = 0$

$$0 = \det(H-\lambda I) = \begin{vmatrix} J-\lambda & h & h & 0 \\ h & -J-\lambda & 0 & h \\ h & 0 & -J-\lambda & h \\ 0 & h & h & J-\lambda \end{vmatrix} = \begin{vmatrix} J-\lambda & h & h & 0 \\ h & -J-\lambda & 0 & h \\ 0 & J+\lambda & -J-\lambda & 0 \\ 0 & h & h & J-\lambda \end{vmatrix}$$

$$= (J-\lambda)\begin{vmatrix} -J-\lambda & 0 & h \\ J+\lambda & -J-\lambda & 0 \\ h & h & J-\lambda \end{vmatrix} - h\begin{vmatrix} h & h & 0 \\ J+\lambda & -J-\lambda & 0 \\ h & h & J-\lambda \end{vmatrix}$$

$$= (J-\lambda)\{(-J-\lambda)[(-J-\lambda)(J-\lambda)] + h[h(J+\lambda) - h(-J-\lambda)]\}$$
$$-h\{h[(-J-\lambda)(J-\lambda)] - h[(J+\lambda)(J-\lambda)]\}$$

$$= (J+\lambda)^2(J-\lambda^2) + 2h^2(J+\lambda)(J-\lambda) + 2h^2(J+\lambda)(J-\lambda)$$

$$= (J+\lambda)^2(J-\lambda)^2 + 4h^2(J+\lambda)(J-\lambda)$$

We see the two roots $(J+\lambda)=0$ and $(J-\lambda)=0$, so divide by $(J+\lambda)(J-\lambda)$:

$$(J+\lambda)(J-\lambda) + 4h^2 = 0 \Rightarrow J^2 - \lambda^2 + 4h^2 = 0 \Rightarrow \lambda = \pm\sqrt{J^2+4h^2}$$

Therefore the four energy eigenvalues are $\pm J, \pm\sqrt{J^2+4h^2}$

The lowest of these is $E_0 = -\sqrt{J^2+4h^2}$ which is the ground state energy.