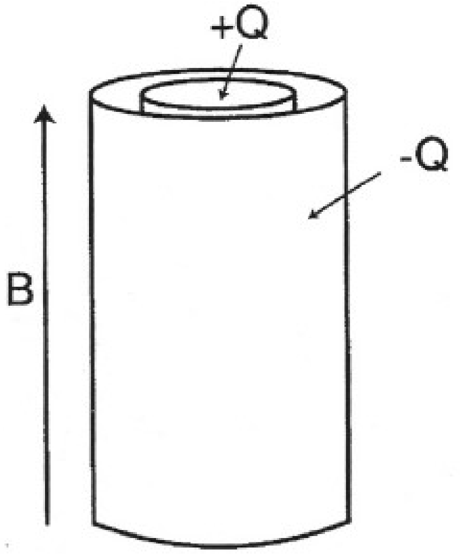
10. *Electricity and Magnetism* (Spring 2004)

Consider a cylindrical capacitor of length L with charge $+Q$ on the inner cylinder of radius a and $-Q$ on the outer cylindrical shell of radius b. The capacitor is filled with a lossless dielectric with dielectric constant equal to 1. The capacitor is located in a region with a uniform magnetic field B, which points along the symmetry axis of the cylindrical capacitor. A flaw develops in the dielectric insulator, and a current flow develops between the two plates of the capacitor. Because of the magnetic field, this current flow results in a torque on the capacitor, which begins to rotate. After the capacitor is fully discharged (total charge on both plates is zero), what is the magnitude and direction of the angular velocity of the capacitor? The moment of inertia of the capacitor (about the axis of symmetry) is I, and you may ignore fringing fields in the calculation.



Let $d\vec{L}$ be the change in angular momentum due to the flow of an infinitesimal amount of charge dq. Then $\vec{L} = \int_0^Q d\vec{L}$.

$$d\vec{L} = \int_0^t \vec{\tau}\, dt = \int_a^b \vec{\tau}(r) \frac{dt}{dr} dr = \int_a^b \vec{r} \times \vec{F}(\vec{r}) \frac{1}{v} dr$$

$$= dq \int_a^b \vec{r} \times (\vec{v} \times \vec{B}) \frac{1}{v} dr = dq \int_a^b r B\, \hat{r} \times (\hat{r} \times \hat{z})\, dr$$

$$= -\frac{1}{2}(b^2 - a^2) B\, dq\, \hat{z}$$

$$\vec{L} = \int_0^Q d\vec{L} = -\frac{1}{2}(b^2 - a^2) B Q \hat{z}$$

$$\vec{L} = I\vec{\omega} \Rightarrow \vec{\omega} = -\frac{1}{2} \frac{QB}{I} (b^2 - a^2) \hat{z}$$