3. *Quantum Mechanics* (Spring 2004)

The normalized wave function of a one-dimensional particle is

$$\psi(x) = N e^{-\kappa x^2/2}$$

for some $\kappa > 0$. N is real and positive.

(a) What is N?

(b) What is the expectation value of x^2?

(c) What is the momentum space wave function $\langle p|\,\psi\rangle$?

(d) What is the expectation value of p^2?

(e) The Hamiltonian is

$$H = \frac{p^2}{2m} + V(x)$$

What is the potential $V(x)$?

a. By normalization $1 = \langle \Psi|\Psi\rangle = \int_{-\infty}^{\infty} N^* N e^{-Kx^2} dx = 2|N|^2 \int_0^\infty e^{-Kx^2} dx$

$= 2|N|^2 \frac{1}{\sqrt{K}} \int_0^\infty e^{-u^2} du = 2|N|^2 \frac{1}{\sqrt{K}} \frac{\sqrt{\pi}}{2} \Rightarrow |N|^2 = \sqrt{\frac{K}{\pi}}$

$\Rightarrow N = \left(\frac{K}{\pi}\right)^{1/4}$ since N is real and positive

b. $\langle\Psi|x^2|\Psi\rangle = \sqrt{\frac{K}{\pi}} \int_{-\infty}^{\infty} x^2 e^{-Kx^2} dx = \sqrt{\frac{K}{\pi}}\, K^{-3/2}\, 2\int_0^\infty u^2 e^{-u^2} du$

$= \frac{1}{K\sqrt{\pi}} \cancel{2}\left(\frac{1}{\cancel{2}}\Gamma\left(\frac{2+1}{2}\right)\right) = \frac{1}{K\sqrt{\pi}}\left(\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right) = \frac{1}{K\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2}\right) = \frac{1}{2K}$

Using the formula $\int_0^\infty x^n e^{-x^2} dx = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)$

c. Recall $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right)$

$\langle p|\Psi\rangle = \int_{-\infty}^{\infty} \langle p|x\rangle\langle x|\Psi\rangle dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} N e^{-Kx^2/2} dx$

$= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{K}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} e^{-Kx^2/2 + ipx/\hbar} dx$

$= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{K}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} e^{-\frac{K}{2}\left[\left(x - \frac{ip}{K\hbar}\right)^2 + \frac{p^2}{K^2\hbar^2}\right]} dx$ (completing the square)

$= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{K}{\pi}\right)^{1/4} e^{-p^2/2K\hbar^2} \int_{-\infty}^{\infty} e^{-\frac{K}{2}u^2} du$

$= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{K}{\pi}\right)^{1/4} e^{-p^2/2K\hbar^2} \sqrt{\frac{2}{K}}\sqrt{\pi} = \frac{1}{\sqrt{\hbar}(K\pi)^{1/4}} e^{-p^2/2K\hbar^2}$

d. $\langle\Psi|p^2|\Psi\rangle = \int_{-\infty}^{\infty} p^2 |\langle p|\Psi\rangle|^2 dp = \frac{1}{\hbar\sqrt{K\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2/K\hbar^2} dp$

$= \frac{1}{\hbar\sqrt{K\pi}} (K\hbar^2)^{3/2} \frac{\sqrt{\pi}}{2} = \frac{1}{2}\hbar^2 K$

e. $-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi = E\Psi \Rightarrow \frac{\partial^2\Psi}{\partial x^2} = \frac{2m(V-E)}{\hbar^2}\Psi$

$\frac{\partial\Psi}{\partial x} = -KxNe^{-Kx^2/2}$ $\quad \frac{\partial^2\Psi}{\partial x^2} = K^2x^2Ne^{-Kx^2/2} - KNe^{-Kx^2/2} = (K^2x^2 - K)\Psi$

$\Rightarrow \frac{2m(V-E)}{\hbar^2} = K^2x^2 - K \quad \Rightarrow V - E = \frac{\hbar^2K^2}{2m}x^2 - \frac{\hbar^2 K}{2m}$

$\Rightarrow V(x) = \frac{\hbar^2K^2}{2m}x^2 + C$ where C is a constant equal to $E - \frac{\hbar^2 K}{2m}$

which makes sense because the ground state of the SHO is gaussian like $\Psi(x)$ and the SHO potential is quadratic.