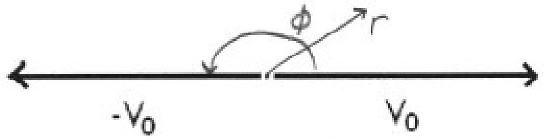
9. *Electricity and Magnetism* (Spring 2004)

Consider the infinite two-dimensional conducting plane depicted in the figure. The right half is maintained at electrostatic potential V_0 while the left half is maintained at potential $-V_0$. What is the potential above the plane?



See Jackson Section 2.11

We solve Laplace's equation in cylindrical coordinates

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

There is no z dependence by symmetry so we use separation of variables and seek solutions of the form $\Phi(r,\phi) = R(r)Q(\phi)$.
(Or you could recall that the solution is $\Phi(r,\phi) = (A + B\ln(r))(C + D\phi)$ when r ranges from 0 to ∞).

$$\nabla^2\Phi = 0 \Rightarrow \frac{Q}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{R}{r^2}\frac{\partial^2 Q}{\partial \phi^2} = 0$$

$$\Rightarrow \frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{Q}\frac{\partial^2 Q}{\partial \phi^2} = 0$$

$$\Rightarrow \frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = \lambda \quad \text{and} \quad \frac{1}{Q}\frac{\partial^2 Q}{\partial \phi^2} = -\lambda$$

by independence of variables

$$\Rightarrow r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = \lambda R \quad \text{and} \quad \frac{\partial^2 Q}{\partial \phi^2} = -\lambda Q$$

$$\Rightarrow \begin{cases} R(r) = Ar^{\sqrt{\lambda}} + Br^{-\sqrt{\lambda}} & \text{and } Q(\phi) = C\sin(\sqrt{\lambda}\phi) + D\cos(\sqrt{\lambda}\phi) \quad (\lambda \neq 0) \\ R(r) = A' + B'\ln(r) & \text{and } Q(\phi) = C' + D'\phi \quad (\lambda = 0) \end{cases}$$

The conditions that $|\Phi(r=\infty)| < \infty$ and $|\Phi(r=0)| < \infty$ imply $A = B = B' = 0$, so the $\lambda \neq 0$ case is excluded.

$$\Rightarrow \Phi(r,\phi) = C' + D'\phi$$

$$\Phi(\phi = 0) = V_0 \Rightarrow C' = V_0 \quad \text{and} \quad \Phi(\phi = \pi) = -V_0 \Rightarrow D' = -\frac{2V_0}{\pi}$$

Therefore $\Phi(r,\phi) = V_0\left(1 - \frac{2}{\pi}\phi\right)$