10. *Electricity and Magnetism* (Spring 2005)

A relativistic charged particle of charge q and rest-mass m_o is in a region of uniform magnetic field $B_o\hat{z}$. At time $t = 0$ the particle has zero velocity along $\hat{z}$ (that is $\beta_z = v_z/c = 0$) and finite transverse speed $\beta_\perp = \beta_o$, with

$$\beta_\perp = \sqrt{v_x^2 + v_y^2}/c$$

Here, x, y, and z are Cartesian coordinates in the lab frame.

(a) What is the value of $\beta_\perp(t)$ for $t > 0$?

(b) What is the angular frequency Ω of rotation (that is, the gyrofrequency)? No need for a calculation, just identify Ω.

(c) Now apply a uniform electric field $E_o\hat{z}$, parallel to $\mathbf{B}$, starting at $t = 0$. Without solving the detailed equations, conclude what happens to the $\beta_\perp$ in part (a). Does it change?

a. Magnetic forces are always perpendicular to the direction of the field, so $\beta_z(t) = 0$. Also, magnetic forces do no work, so $\beta(t) = \beta(0) \Rightarrow \beta_\perp(t) = \beta_o$

b. Basically we just have to use the relativistic momentum $\vec{p} = \gamma m_o \vec{v}$. The force is $\vec{F} = q\vec{v}\times\vec{B} = qc\vec{\beta}\times\vec{B}$

$$\Rightarrow \vec{F} = qcB_o(\beta_y\hat{x} - \beta_x\hat{y})$$

$$\vec{F} = \frac{d\vec{p}}{dt} \Rightarrow \frac{dp_x}{dt} = qcB_o\beta_y \quad \text{and} \quad \frac{dp_y}{dt} = -qcB_o\beta_x$$

$$\gamma \text{ is constant} \Rightarrow \frac{dv_x}{dt} = \frac{qcB_o}{\gamma m_o}\beta_y \quad \text{and} \quad \frac{dv_y}{dt} = -\frac{qcB_o}{\gamma m_o}\beta_x$$

$$\Rightarrow \frac{d^2v_x}{dt^2} = \frac{qB_o}{\gamma m_o}\frac{dv_y}{dt} = -\left(\frac{qB_o}{\gamma m_o}\right)^2 v_x$$

$$\Rightarrow v_x(t) = A\sin\left(\frac{qB_o}{\gamma m_o}t + \phi\right)$$

$$\Rightarrow \Omega = \frac{qB_o}{\gamma m_o}$$

c. There is the constraint $\beta^2 = \beta_\perp^2 + \beta_z^2 \leq 1$, so as the electric field accelerates β_z toward 1, $\beta_\perp$ must approach 0.