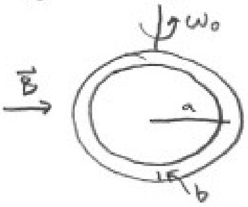
12. *Electricity and Magnetism* (Spring 2005)

A thin copper ring (conductivity σ, density ρ) is suspended so it can rotate freely about one diameter. There is a uniform magnetic field $\mathbf{B}$ perpendicular to the axis of rotation. At time $t = 0$ the ring is set rotating with frequency ω_o. Calculate the time it takes the frequency to decrease to $1/e$ of its original value, assuming the energy goes into Joule heating.



$$\mathcal{E} = -\frac{\partial \Phi_B}{\partial t} = -\frac{\partial}{\partial t}\left(\int_S \vec{B}\cdot d\vec{a}\right) = -\frac{\partial}{\partial t}\left(\int_S B\,da\cos(\omega t)\right)$$

$$= -B\frac{\partial}{\partial t}\left(\cos(\omega t)\int_S da\right) = +\pi a^2 B\omega\sin(\omega t)$$

$$\frac{dE_J}{dt} = P = \frac{\mathcal{E}^2}{R} \quad \text{where } R = \rho_R\frac{l}{A} = \frac{1}{\sigma}\frac{2\pi a}{\pi b^2} = \frac{2a}{\sigma b^2}$$

$$\Rightarrow \frac{dE_J}{dt} = \frac{\sigma b^2}{2a}\pi^2 a^4 B^2\omega^2\sin^2(\omega t) = \frac{1}{2}\pi^2\sigma a^3 b^2 B^2\omega^2\sin^2(\omega t)$$

We must assume ω is large enough that we can use the average power

$$\left\langle \frac{dE_J}{dt}\right\rangle = \frac{1}{4}\pi^2\sigma a^3 b^2 B^2\omega^2$$

Now, the Kinetic energy is $T = \frac{1}{2}I\omega^2$ where $I = \int r^2 dm$

with $r = a\sin(\theta)$, $m(\theta) = (\pi b^2\rho)(a\theta) \Rightarrow \frac{dm}{d\theta} = \pi a b^2\rho$

So $I = 2\int_0^\pi a^2\sin^2(\theta)\pi a b^2\rho\, d\theta = 2\pi a^3 b^2\rho\int_0^\pi \sin^2(\theta)d\theta = \pi^2 a^3 b^2\rho$

Therefore $T = \frac{1}{2}I\omega^2 = \frac{1}{2}\pi^2 a^3 b^2\rho\omega^2$

$$\Rightarrow \frac{dT}{dt} = \pi^2 a^3 b^2\rho\omega\frac{d\omega}{dt}$$

By conservation of energy, the Kinetic energy is decreasing at the instantaneous average rate that Joule heating energy is increasing.

$$\Rightarrow \frac{dT}{dt} = -\left\langle\frac{dE_J}{dt}\right\rangle \Rightarrow \cancel{\pi^2}\cancel{a^3}\cancel{b^2}\rho\omega\frac{d\omega}{dt} = -\frac{1}{4}\cancel{\pi^2}\sigma\cancel{a^3}\cancel{b^2}B^2\omega^2$$

$$\Rightarrow \frac{d\omega}{dt} = -\frac{\sigma B^2}{4\rho}\omega$$

$$\Rightarrow \omega(t) = \omega_o e^{-\frac{\sigma B^2}{4\rho}t}$$

So the frequency reaches $\frac{\omega_o}{e}$ when $t = \frac{4\rho}{\sigma B^2}$