7. *Statistical Mechanics and Thermodynamics* (Spring 2005)

A (nearly) ideal gas with a temperature T and pressure P contains atoms of mass M that are either in the ground state or in the first excited state. An atom that returns to the ground state from the first excited state emits a photon of frequency f_o. For a stationary observer observing the spectral line emitted by a *moving* atom, this frequency is shifted by the Doppler effect to

$$f(v_\parallel) = f_o(1 + v_\parallel/c)$$

where c is the velocity of light and $v_\parallel$ is the projection of the velocity of the atom on the line of sight from the observer to the atom.

(a) What is the statistical distribution $P(f)$ of the frequency of the spectal line? Assume the atoms obey the Maxwell-Boltzmann distribution.

(b) Obtain from $P(f)$ the contribution by the Doppler effect to the width $\sqrt{\langle (f - f_o)^2 \rangle}$ of the spectral line. Can you think of a way this effect could be exploited in the study of stellar atmospheres?

(c) The *natural* line shape $P(f)$ of an atomic spectral line is, according to quantum mechanics, given by

$$P(f) \sim \frac{1}{(f - f_o)^2 + \tau^{-2}}$$

where τ is the *lifetime* of the excited state. For atoms in a dense gas, the actual lifetime of the excited state is not intrinsic, but instead determined by the time interval between successive collisions between atoms. Let the cross section of an atom equal σ. Obtain an expression for τ in terms of σ, the pressure P and the temperature T. Under which conditions will this "collisional" broadening of the spectral line dominate over the Doppler broadening as computed under (b)?

a. We start from $P_f(f)\,df = P_{v_\parallel}(v_\parallel)\,dv_\parallel$. Then $f = f_o(1 + v_\parallel/c)$

$\Rightarrow f - f_o = \frac{f_o}{c} v_\parallel \Rightarrow v_\parallel = \frac{c}{f_o}(f - f_o)$

Therefore $P_f(f) = P_{v_\parallel}\left(\frac{c}{f_o}(f-f_o)\right)\frac{dv_\parallel}{df} = P_{v_\parallel}\left(\frac{c}{f_o}(f-f_o)\right)\frac{c}{f_o}$

By the Maxwell-Boltzmann distribution, $P_{v_\parallel}(v_\parallel)\,dv_\parallel = \left(\frac{m}{2\pi KT}\right)^{1/2} e^{-m v_\parallel^2/2KT}\,dv_\parallel$

$\Rightarrow P_f(f) = \frac{c}{f_o}\left(\frac{m}{2\pi KT}\right)^{1/2} e^{-m\left(\frac{c}{f_o}\right)^2 (f-f_o)^2/2KT}$

b. The equation for a Gaussian is $G(x) = A e^{-(x-x_o)^2/2\sigma^2}$

$\Rightarrow m\left(\frac{c}{f_o}\right)^2/2KT = \frac{1}{2\sigma^2} \Rightarrow \sigma^2 = \frac{KT}{m}\left(\frac{f_o}{c}\right)^2 \Rightarrow \sigma = f_o\sqrt{\frac{KT}{mc^2}}$

So the linewidth can be used to determine the temperature of a stellar atmosphere.

c. $n\bar{v}\sigma$ particles scatter per unit time off one particle

$\Rightarrow \tau^{-1} = n\bar{v}\sigma$ and for an ideal gas $pV = NKT \Rightarrow n = \frac{p}{KT}$

$\Rightarrow \tau^{-1} = \frac{p}{KT}\bar{v}\sigma \Rightarrow \tau = \frac{KT}{p\bar{v}\sigma}$ and $\bar{v} = \sqrt{\frac{8KT}{\pi M}}$

$\Rightarrow \tau = \frac{KT}{p\sigma}\sqrt{\frac{\pi M}{8KT}} = \frac{\sqrt{M\pi/8}}{p\sigma}(KT)^{1/2}$

The broadening is largest when $P(f)$ is large when f is far from f_o, which happens when τ^{-2} is large, so τ is small, so either T is small or p is large.