1. *Quantum Mechanics* (Spring 2006)

An electron is at rest in a constant magnetic field pointing along the z-direction. The Hamiltonian is

$$H = -\boldsymbol{\mu} \cdot \mathbf{B} = g\mu_0 \frac{\mathbf{s}}{\hbar} \cdot \mathbf{B}$$

where $\mathbf{B} = B_0 \hat{\boldsymbol{n}}_z$. Since the electron is at rest, you can treat this as a two-state system. Let $|\psi_\pm\rangle$ be the eigenstates of s_z with eigenvalues $\pm\frac{\hbar}{2}$ respectively.

(a) What are the eigenstates of the Hamiltonian in terms of $|\psi_\pm\rangle$, and what is the energy difference between them?

(b) At time $t = 0$ the electron is in an eigenstate of s_x with eigenvalue $+\hbar/2$. What is $|\psi(0)\rangle$ in terms of $|\psi_\pm\rangle$? Calculate $|\psi(t)\rangle$ for any later time t in terms of these same two states.

(c) For the state you calculated in part (b), what are the expectation values of the three components of the spin at any time t?

a. $H = g\mu_0 \frac{\vec{s}}{\hbar} \cdot \vec{B} = \frac{g}{2} \mu_0 \vec{\sigma} \cdot \vec{B} = \frac{g}{2} \mu_0 \sigma_z B_0 \cong \mu_0 \sigma_z B_0$

The eigenvalues of σ_z are ± 1, so the eigenvalues of H are $\pm \mu_0 B_0 \Rightarrow \Delta E = \mu_0 B_0 - (-\mu_0 B_0) = 2\mu_0 B_0$

b. $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow |\psi_{x+}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ since $\sigma_x |\psi_{x+}\rangle = (+1)|\psi_{x+}\rangle$

So $|\psi(0)\rangle = |\psi_{x+}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} |\psi_{z+}\rangle + \frac{1}{\sqrt{2}} |\psi_{z-}\rangle$

$|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle = \frac{1}{\sqrt{2}} e^{-i\mu_0 B_0 t/\hbar} |\psi_{z+}\rangle + \frac{1}{\sqrt{2}} e^{i\mu_0 B_0 t/\hbar} |\psi_{z-}\rangle$

c. $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \pm \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix} \Rightarrow a = \pm b$

$\Rightarrow |\psi_{x+}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $|\psi_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \Rightarrow \pm \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -ib \\ ia \end{pmatrix} \Rightarrow b = \pm ia$

$\Rightarrow |\psi_{y+}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $|\psi_{y-}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \pm \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix} \Rightarrow a = 0$ or $b = 0$

$\Rightarrow |\psi_{z+}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\psi_{z-}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\langle s_x \rangle = \frac{\hbar}{2} |\langle \psi_{x+} | \psi(t) \rangle|^2 + \left(-\frac{\hbar}{2}\right) |\langle \psi_{x-} | \psi(t) \rangle|^2$

$= \frac{\hbar}{2} |\cos(\mu_0 B_0 t/\hbar)|^2 + \left(-\frac{\hbar}{2}\right) |-\sin(\mu_0 B_0 t/\hbar)|^2$

$= \frac{\hbar}{2} \cos(2\mu_0 B_0 t/\hbar)$

$\langle s_y \rangle = \frac{\hbar}{2} |\langle \psi_{y+} | \psi(t) \rangle|^2 + \left(-\frac{\hbar}{2}\right) |\langle \psi_{y-} | \psi(t) \rangle|^2$

$= \frac{\hbar}{2} \left| \frac{1}{2} \left(e^{-i\mu_0 B_0 t/\hbar} + i e^{i\mu_0 B_0 t/\hbar} \right) \right|^2 + \left(-\frac{\hbar}{2}\right) \left| \frac{1}{2} \left(e^{-i\mu_0 B_0 t/\hbar} - i e^{i\mu_0 B_0 t/\hbar} \right) \right|^2$

$= \frac{\hbar}{2} \frac{1}{4} \left[(\cos - \sin)^2 + (-\sin + \cos)^2 \right] + \left(-\frac{\hbar}{2}\right) \frac{1}{4} \left[(\cos + \sin)^2 + (-\sin - \cos)^2 \right]$

$= \frac{\hbar}{2} \frac{1}{2} (\cos - \sin)^2 - \frac{\hbar}{2} \frac{1}{2} (\cos + \sin)^2$

$= -\frac{\hbar}{2} \frac{1}{2} 4 \sin\cos = -\frac{\hbar}{2} \sin(2\mu_0 B_0 t/\hbar)$

$\langle s_z \rangle = \frac{\hbar}{2} |\langle \psi_{z+} | \psi(t) \rangle|^2 + \left(-\frac{\hbar}{2}\right) |\langle \psi_{z-} | \psi(t) \rangle|^2 = \frac{\hbar}{2} \left(\frac{1}{2}\right) - \frac{\hbar}{2} \left(\frac{1}{2}\right) = 0$