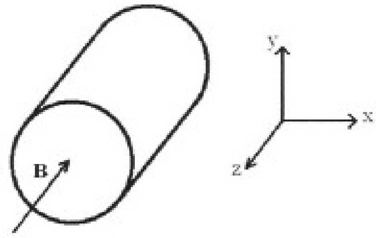
11. *Electricity and Magnetism* (Spring 2006)

Consider a long solid cylinder made of uniform resistive material. The cylinder is in a region in which there is an applied magnetic field that is uniform and is directed along the axis of the cylinder. The magnetic field is time-dependent and it is oscillating with angular frequency ω: $\mathbf{B}(t) = B_z \cos\omega t\hat{\mathbf{z}}$. The length of the cylinder is L and its radius is R ($R \ll L$). The resistivity of the cylinder material is ρ.



(a) Calculate the current density $\mathbf{j}(t)$ in the volume of the cylinder. Assume initially that you can ignore the self-inductance of the cylinder. Ignore end effects and the Hall effect.

(b) For large values of ω the effect of self-inductance cannot be ignored. Calculate the correction to the current density $\Delta\mathbf{j}(t)$ due to the self-inductance of the cylinder in next order of ω.

(c) Give the condition on ω such that the self-inductance of the cylinder can be ignored.

a. $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \int_c \vec{E}\cdot d\vec{l} = -\int_s \frac{\partial\vec{B}}{\partial t}\cdot d\vec{a} = -\int_s -\omega B_z \sin(\omega t)\hat{z}\cdot d\vec{a}$

$\Rightarrow 2\pi r E = \omega B_z \sin(\omega t)\pi r^2 \Rightarrow \vec{E} = \frac{1}{2}\omega B_z \sin(\omega t) r\,\hat{\phi}$

$\vec{J} = \sigma\vec{E} = \frac{1}{\rho}\vec{E} \Rightarrow \vec{J} = \frac{1}{2\rho}\omega B_z \sin(\omega t) r\hat{\phi}$

b. First we find the correction to the magnetic field due to all the solenoids outside radius r. $\vec{B}_{sol}(r,t) = \mu_0 K(r,t)$

$d\vec{K}(r) = \vec{J}(r)dr$ and $d^2\vec{I}(r) = d\vec{K}(r)dz = \vec{J}(r)\,dr\,dz \Rightarrow d\vec{B}_{sol}(r,t) = \mu_0\vec{J}(r,t)dr$

$\Delta\vec{B}(r,t) = \int_r^R d\vec{B}_{sol}(r',t) = \frac{\mu_0}{2\rho}\omega B_z \sin(\omega t)\hat{\phi}\int_r^R r'dr'$

$= \frac{\mu_0}{2\rho}\omega B_z \sin(\omega t)\hat{\phi}\left(\frac{1}{2}(R^2-r^2)\right) = \frac{\mu_0}{4\rho}\omega B_z \sin(\omega t)(R^2-r^2)\hat{\phi}$

$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \int_c \vec{E}\cdot d\vec{l} = -\int_s \frac{\partial\vec{B}}{\partial t} = -\int_0^{2\pi}\int_0^r \frac{\partial\vec{B}}{\partial t} r'dr'd\theta'$

$\Rightarrow 2\pi r\Delta E = -2\pi\int_0^r \frac{\partial}{\partial t}(\Delta B)r'dr'$

$\Rightarrow \Delta E = -\frac{1}{r}\int_0^r \frac{\mu_0}{4\rho}\omega^2 B_z \cos(\omega t)(R^2-r'^2)r'dr'$

$= -\frac{1}{r}\frac{\mu_0}{4\rho}\omega^2 B_z \cos(\omega t)\left(\frac{1}{2}r^2R^2 - \frac{1}{4}r^4\right)$

$= -\frac{\mu_0}{4\rho}\omega^2 B_z \cos(\omega t)\left(\frac{1}{2}rR^2 - \frac{1}{4}r^3\right)$

$\Rightarrow \Delta\vec{J} = \frac{1}{\rho}\Delta\vec{E} = -\frac{\mu_0}{4\rho^2}\omega^2 B_z \cos(\omega t)\left(\frac{1}{2}rR^2 - \frac{1}{4}r^3\right)\hat{\phi}$

c. $\left|\frac{\Delta\vec{J}}{\vec{J}}\right| \ll 1 \Leftrightarrow \frac{\frac{\mu_0}{4\rho^2}\omega^2 B_z}{\frac{1}{2\rho}\omega B_z} \ll 1 \Leftrightarrow \frac{\mu_0}{\rho}\omega \ll 1 \Leftrightarrow \omega \ll \frac{\rho}{\mu_0}$