6. *Statistical Mechanics and Thermodynamics* (Spring 2006)

Consider a gas of relativistic, conserved bosons. The relation between energy and momentum is

$$E = |\mathbf{p}|c$$

(a) Derive the condition for Bose-Einstein condensation in three dimensions.

(b) Does Bose-Einstein condensation occur in two dimensions? Justify your answer.

(c) What is the highest dimension for which Bose-Einstein condensation does not occur?

The simplest definition of T_c is the minimum temperature for which all particles in the system are expected to be in excited states. Our strategy:

1. Find the density of states
2. Integrate occupancy times density of states to get the total number of particles in excited states N_e (since $\epsilon=0$ for ground state, they aren't counted in this integral because $\rho(0)=0$).
3. Maximize N_e by setting $\mu=0$ so the minimum temperature comes out.
4. Set $N_e=N$ and solve for $T=T_c$.

a. $\epsilon = pc = \hbar k c = \left(\frac{\hbar\pi c}{L}\right)n \Rightarrow n = \left(\frac{L}{\hbar\pi c}\right)\epsilon \Rightarrow dn = \left(\frac{L}{\hbar\pi c}\right)d\epsilon$

$$\rho(\epsilon) = \frac{1}{8}4\pi n^2 dn = \frac{\pi}{2}\left(\frac{L}{\hbar\pi c}\right)^3 \epsilon^2 d\epsilon = \frac{V}{2\pi^2}\frac{\epsilon^2}{(\hbar c)^3}d\epsilon$$

$$\begin{aligned}
N_e &= \int_0^\infty f(\epsilon)\rho(\epsilon)\,d\epsilon \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\int_0^\infty \frac{\epsilon^2}{e^{\beta(\epsilon-\mu)}-1}\,d\epsilon \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\int_0^\infty \frac{\epsilon^2}{e^{\beta(\epsilon-\mu)}}\frac{1}{1-e^{-\beta(\epsilon-\mu)}}\,d\epsilon \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\int_0^\infty \frac{\epsilon^2}{e^{\beta(\epsilon-\mu)}}\sum_{l=0}^\infty e^{-\beta l(\epsilon-\mu)}\,d\epsilon \qquad (\text{must have } \epsilon>\mu \text{ for } \bar{n}>0) \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\int_0^\infty \epsilon^2 \sum_{l=1}^\infty e^{-\beta l(\epsilon-\mu)}\,d\epsilon \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\sum_{l=1}^\infty\left[e^{\beta l\mu}\int_0^\infty \epsilon^2 e^{-\beta l\epsilon}\,d\epsilon\right] \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\sum_{l=1}^\infty\left[e^{\beta l\mu}\left(\frac{1}{\beta l}\right)^3\int_0^\infty x^2 e^{-x}\,dx\right] \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\frac{2!}{\beta^3}\sum_{l=1}^\infty \frac{e^{\beta l\mu}}{l^3} \\
&\xrightarrow{\mu\to 0} \frac{V}{\pi^2}\frac{1}{(\hbar c\beta)^3}\sum_{l=1}^\infty \frac{1}{l^3} \quad \text{and} \quad \sum_{l=1}^\infty \frac{1}{l^3} \equiv \zeta(3)
\end{aligned}$$

$$\begin{aligned}
N_e = N \Rightarrow \frac{1}{\beta^3} &= \frac{N}{V}\pi^2\frac{(\hbar c)^3}{\zeta(3)} \\
\Rightarrow \frac{1}{\beta} &= \hbar c\left(\pi^2\frac{N}{V}/\zeta(3)\right)^{1/3} \\
\Rightarrow T_c &= \frac{\hbar c}{k}\left(\pi^2\frac{N}{V}/\zeta(3)\right)^{1/3}
\end{aligned}$$

b. In 2D, $\rho(\epsilon)d\epsilon = \frac{1}{4}2\pi n\,dn = \frac{\pi}{2}\left(\frac{L}{\hbar\pi c}\right)^2\epsilon\,d\epsilon$ gives $\zeta(2)$ with a similar procedure and $\zeta(2)$ converges so everything is fine and condensation does occur.

c. In 1D, $\rho(\epsilon)d\epsilon = dn = \frac{L}{\hbar\pi c}d\epsilon$ gives $\zeta(1)$ with a similar procedure, but $\zeta(1)$ diverges, so the resulting T_c is $T_c=0$, so BEC does not occur in 1D, making 1 the highest dimension for which BEC does not occur.