Gravitomagnetism

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1 Qualitative Introduction

What is gravitomagnetism? It is an effect that looks like magnetism, but is caused by the gravitational force. Its field is generated by currents of matter flow and its force law has the same form as the Lorentz force law: $\mathbf{v} \times \mathbf{B}$. There is not yet conclusive evidence for it[1] but there probably will be soon with Gravity Probe B. The best example is probably the phenomenon of "frame dragging" when falling into a spinning black hole. As you fall toward it, you have an inward radial velocity which reacts with the gravitomagnetic field to cause you to be pulled along with the rotation of the black hole. This may not be the way that you think of EM magnetism, but it is the same. If a positive charge is moving toward a wire carrying a negative current (so the force between the two is attractive like in gravity), the force acts to drag the positive charge along with the frame of the negative current.[2]

2 Conceptual Points

Our primary goal was to convince ourselves that gravitation is basically the same as electromagnetism. There are three obvious differences

(a) Gravity is weaker, but that is just a scale factor

(b) In gravity, like attracts like, but that is just a negative sign

(c) Gravitational fields act as sources for themselves, but that is just a nonlinear correction [3]

So we thought, start by comparing Newton's Law of Gravity and Coulomb's Law of Electricity. They are the same up to a negative scale factor. Then think of magnetism as a consequence in both cases. This can be argued as a result of local Lorentz invariance in special relativity. Then there is the nonlinearity in gravitation. If you remove the nonlinearity by linearizing the equations, you would expect that you would be left with Maxwell's equations. However, you don't get Maxwell's equations exactly, as we will soon see.

Our secondary goal was to emphasize that the linearized equations can be thought of as more fundamental than Einstein's equations. The dictionary defines fundamental by "serving as an original or generating source." As is well known, the full Einstein's equation accounts for gravitational feedback, and feedback is not original or generating, it is a consequence of the fundamental laws.[4] In Einstein's equation,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$

we have all the energy *except* gravitational energy on the right hand side. If you wrote an equation with all energy *including* gravitational energy, then you should have a linearized form of Einstein's equations. Another way to think of this is by analogy with Maxwell's equations. The linearized equations are like Maxwell's equations in vacuum, whereas the full Einstein equation is like Maxwell's equations in matter because both have an infinite feedback loop. For EM, the fields set up a polarization, which influences the fields, and so on. For Gravity, the fields contain energy, which influences the fields, and so on. It may be more enlightening to understand the underlaying law of the force stripped bare of its feedback mechanisms, despite the fact that the feedback is inevitable.

Can we be sure that the linearized equations don't drop some nonlinear terms that are *not* associated with feedback? Carroll writes on page 299:

[1] Mach's Principle From Newton's Bucket to Quantum Gravity Page 390

[2] If you are at a constrained radius around the spinning black hole then you will not feel a gravitomagnetic force.

[3] Note that the fact that all matter has positive mass is not a property of gravity itself, but a property of the universe.

[4] I also like the example of what laws you would have to enter into a universe simulator code.

> In fact, starting with a theory of spin-2 gravitons and requiring some simple properties provides a nice way to *derive* the full Einstein's equation of general relativity. Imagine starting with the Lagrangian (7.9) for the symmetric tensor $h_{\mu\nu}$, but now imagining that this "really is" a physical field propagating in Minkowski spacetime rather than a perturbation to a dynamical metric. (This Lagrangian doesn't include couplings to matter, but it is straightforward to do so.) Now make the additional demand that $h_{\mu\nu}$ couple to its own energy-momentum tensor (discussed below), as well as to the matter energy-momentum tensor. This induces higher-order nonlinear terms in the action and consequently induces additional "energy-momentum" terms of even higher order. By repeating this process, an infinite series of terms is introduced, but the series can be summed to a simple expression, perhaps because you already know the answer—the Einstein-Hilbert action (possibly with some higher-order terms).

If there are higher order terms, then that could mean that there are complicated nonlinear effects not due entirely to feedback.[5]

3 Deriving Einstein's Equation

Let's quickly review the derivation of Einstein's equation given in Carroll.

$$\begin{aligned}
\text{Poisson equation:} \qquad & \nabla^2\Phi = 4\pi G\rho \\
\text{Suggests form:} \qquad & [\nabla^2 g]_{\mu\nu} \propto T_{\mu\nu} \\
\text{Only appropriate tensor is } R_{\mu\nu}\text{:} \qquad & R_{\mu\nu} = \kappa T_{\mu\nu} \\
\text{Enforce } \nabla^\mu T_{\mu\nu} = 0\text{:} \qquad & R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \kappa T_{\mu\nu} \\
\text{Compare to Poisson:} \qquad & R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G T_{\mu\nu}
\end{aligned}$$

The only reason we show this is to emphasize the fact that the derivation does little to enlighten us about the mechanism behind it.

4 Parallelism of Equations

$$\text{Lorentz Force Law} \longleftrightarrow \text{Geodesic Equation}$$

$$\text{Maxwell's Equations} \longleftrightarrow \text{Einstein's Equation}$$

5 Energy-Momentum Tensor

The energy momentum tensor is actually a *rank two tensor field* defined for each point in spacetime. The 00 component of the energy momentum tensor contains all forms of energy except those associated with gravity. Thus it excludes gravitational potential energy (the energy of the gravitoelectric field), the energy of the gravitomagnetic field, and the energy in gravitational radiation, plus anything else which is not listed here. The $0i$ components describe the momentum density. The ii components describe the pressure. The ij $(i \neq j)$ components describe the viscosity. [6]

[5] See also Misner Thorne and Wheeler page 436 for further references.
[6] From Wikipedia "Stress-Energy Tensor"

6 The Linearized Geodesic Equation

We will linearize the geodesic equation and find the analog of the Lorentz force in terms of the potentials. In order to linearize, we linearize the metric by assuming it is of the form $\bar{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu}$ is a small perturbation. From this we see that the linear approximation will only be valid for small deviations from flat space, which is equivalent to weak fields. For illustrative purposes, we reparameterize the components of $h_{\mu\nu}$ in terms of new symbols.

$$h_0 0 = -2\Phi \quad h_{0i} = w_i \quad h_{ij} = 2s_{ij} - 2\Psi\delta_{ij}$$

With these definitions, the linearized metric looks like:

$$\bar{g}_{\mu\nu} = \begin{pmatrix} -(1+2\Phi) & w_1 & w_2 & w_3 \\ w_1 & 1+2s_{11}-2\Psi & 2s_{12} & 2s_{13} \\ w_2 & 2s_{21} & 1+2s_{22}-2\Psi & 2s_{23} \\ w_3 & 2s_{31} & 2s_{32} & 1+2s_{33}-2\Psi \end{pmatrix}$$

Just what do these new symbols mean physically? w_i is analogous to the magnetic vector potential and both Φ and Ψ are analogous to the electric potential in different ways.

We will pre-compute three Christoffel symbols for future reference. Note that any Latin index can be raised or lowered without affecting the result since the spatial part of the Minkowski metric is the Kronecker Delta. However, we will try to follow the natural course without arbitrarily lowering indices.

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)$$

$$\simeq \frac{1}{2} \eta^{\lambda\sigma} \left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\right)$$

$$\Gamma^i_{00} = \frac{1}{2}\eta^{il}\left(\partial_0 h_{0l} + \partial_0 h_{l0} - \partial_l h_{00}\right) = \partial^i\Phi + \partial_0 w^i$$

$$\Gamma^i_{j0} = \frac{1}{2}\eta^{il}\left(\partial_j h_{0l} + \partial_0 h_{lj} - \partial_l h_{j0}\right) = \frac{1}{2}\left(\partial_j w^i - \partial^i w_j + \partial_0 h^i{}_j\right)$$

$$\Gamma^i_{jk} = \frac{1}{2}\eta^{il}\left(\partial_j h_{kl} + \partial_k h_{lj} - \partial_l h_{jk}\right) = \frac{1}{2}\left(\partial_j h_k{}^i + \partial_k h^i{}_j - \partial^i h_{jk}\right)$$

Now observe that for a massive particle

$$p^\mu = m\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\lambda} \quad \text{(if we choose } \lambda = \frac{\tau}{m}\text{)}$$

$$\Rightarrow p^0 = \frac{dx^0}{d\lambda} = c\frac{dt}{d\lambda} = \frac{E}{c} \Rightarrow \frac{d\lambda}{dt} = \frac{c^2}{E}$$

$$\Rightarrow p^i = \frac{dx^i}{d\lambda} = \frac{dx^i}{d\lambda}\frac{d\lambda}{dt}\frac{dt}{d\lambda} = \frac{dx^i}{dt}\frac{dt}{d\lambda} = p^0 v^i/c$$

In GR we always say that particles follow geodesics, but it is entirely equivalent to say that the particles are being forced in flat spacetime, as long as you handle the case of massless particles properly. So here is a very basic and important concept, which is that the geodesic equation can be simply manipulated to give you a force-based perspective. Starting with the Geodesic Equation:

$$\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\rho\sigma}p^\rho p^\sigma = 0$$

$$\frac{dp^\mu}{d\lambda}\frac{d\lambda}{dt} = -\frac{d\lambda}{dt}\Gamma^\mu_{\rho\sigma}p^\rho p^\sigma$$

$$\frac{dp^\mu}{dt} = -\frac{c^2}{E}\Gamma^\mu_{\rho\sigma}p^\rho p^\sigma$$

Now lets just look at the spatial components.

$$\frac{dp^i}{dt} = -\frac{c^2}{E}\left(\Gamma^i_{00}p^0p^0 + 2\Gamma^i_{j0}p^jp^0 + \Gamma^i_{jk}p^jp^k\right)$$

$$\frac{dp^i}{dt} = -\frac{c^2}{E}\left(\Gamma^i_{00}p^0p^0 + 2\Gamma^i_{j0}p^0p^0v^j/c + \Gamma^i_{jk}p^0p^0v^jv^k/c^2\right)$$

$$\frac{dp^i}{dt} = -E\left(\Gamma^i_{00} + 2\Gamma^i_{j0}v^j/c + \Gamma^i_{jk}v^jv^k/c^2\right)$$

As we insert the Christoffel symbols, we will lower the index i (by applying η_{ii} to both sides) in order to be able to see the connection to electromagnetism.

$$\frac{dp_i}{dt} = -E\left(\partial_i\Phi + \partial_0 w_i + (\partial_j w_i - \partial_i w_j + \partial_0 h_{ij})v^j/c + \frac{1}{2}(\partial_j h_{ki} + \partial_k h_{ij} - \partial_i h_{jk})v^jv^k/c^2\right)$$

Now let's focus on the two spatial derivatives of w. It is hard to derive this procedurally, but let's try defining a quantity $\mathbf{H} = \boldsymbol{\nabla}\times\mathbf{w}$, so $H^k = \epsilon^{kmn}\partial_m w_n$. Then look at the expression

$$\begin{aligned}(\mathbf{v}\times\mathbf{H})_i = \epsilon_{ijk}v^jH^k &= \epsilon_{ijk}v^j\epsilon^{kmn}\partial_m w_n \\ = \epsilon_{kij}\epsilon^{kmn}\partial_m w_n v^j &= (\delta^m_i\delta^n_j - \delta^n_i\delta^m_j)\partial_m w_n v^j \\ &= \partial_i w_j v^j - \partial_j w_i v^j\end{aligned}$$

Therefore we can write

$$\frac{dp_i}{dt} = E\left(G_i + \left(\frac{\mathbf{v}}{c}\times\mathbf{H}\right)_i - \partial_0 h_{ij}v^j/c - \frac{1}{2}(\partial_j h_{ki} + \partial_k h_{ij} - \partial_i h_{jk})v^jv^k/c^2\right)$$

where

$$G_i = -\partial_i\Phi - \partial_0 w_i \quad \text{and} \quad H_i = \epsilon_{ijk}\partial^j w^k$$

or

$$\mathbf{G} = -\boldsymbol{\nabla}\Phi - \frac{\partial\mathbf{w}}{\partial t} \quad \text{and} \quad \mathbf{H} = \boldsymbol{\nabla}\times\mathbf{w}$$

which look exactly like the equations for the electromagnetic fields in terms of the electromagnetic potentials

$$\mathbf{E} = -\boldsymbol{\nabla}V - \frac{\partial\mathbf{A}}{\partial t} \quad \text{and} \quad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$$

Notice that all the other terms besides these two have two time derivatives (since a velocity factor counts as a time derivative). So if we approximate the system to be slowly changing so the these terms cancel, we have exactly the electromagnetic case. However, we have not even started to look at Einstein's equation, so these are not at all analogs of Maxwell's equations. It will turn out that there are similarities between the gravitational field equations and Maxwell's equations, but they are not entirely analagous.

7 The Linearized Einstein's Equation

After linearizing Einstein's equation, we will see the analogs of Maxwell's equations in terms of the potentials. The derivation is long because you have to calculate the connection coefficients, the Riemann tensor, the Ricci tensor, and the Ricci scalar. So we are just going to display the results in the transverse gauge.[7]

[7] Question: Does it make sense that we are double-linearizing? i.e. Linearizing both the geodesic equation and Einstein's equation.

$$G_{00} = 2\nabla^2\Psi = 8\pi G T_{00} = 8\pi G\rho^{(m)}$$
$$G_{0j} = -\frac{1}{2}\nabla^2 w_j + 2\partial_0\partial_j\Psi = 8\pi G T_{0j} = -8\pi G J_j^{(m)}$$
$$G_{ij} = (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) - \partial_0\partial_{(i}w_{j)} + 2\delta_{ij}\partial_0^2\Psi - \Box s_{ij} = 8\pi G T_{ij}$$

The negative sign in the right hand side of the second line is due to the fact that the energy-momentum tensor is defined with raised indices. We can replace the fundamental constants by comparing Coulomb's Law and Newton's Law of Gravity

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2} \qquad F = G\frac{m_1 m_2}{r^2}$$
$$\Rightarrow \epsilon_0' = \frac{1}{4\pi G} \qquad \text{and} \qquad \mu_0' = \frac{1}{\epsilon_0' c^2} = \frac{4\pi G}{c^2}$$

Inserting these relations gives from the first two equations,[8]

$$\nabla^2\Psi = \frac{\rho^{(m)}}{\epsilon_0'} \qquad \text{and} \qquad \nabla^2\left(\frac{\mathbf{w}}{4c^2}\right) = \mu_0'\mathbf{J}^{(m)} + \frac{1}{c^2}\frac{\partial\boldsymbol{\nabla}\Psi}{\partial t}$$

which are analogous to

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \qquad \text{and} \qquad \nabla^2\mathbf{A} = -\mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial\boldsymbol{\nabla}V}{\partial t}$$

The sign differences make sense because the electric force between like-charged objects is repulsive, whereas in gravity it is attractive. The Wikipedia article "Gravitomagnetism" suggests that the factor of 4 is due to the fact that the graviton has spin 2.[9] We know that these equations must be true approximations, so if we forget about the third equation it seems like everything worked out perfectly, but there is one major complication, can you see it?

8 Interesting Approximation

The problem is that the linearized Einstein's equation used Ψ as the analog to V whereas the linearized geodesic equation used Φ as the analog to V. The only way this would reduce to Maxwell's equations is if they are the same. How can we even tell if they are? The third equation is for relating the two quantities Φ and Ψ, so lets take a look at it.

Assume linearized Einstein's equation in the transverse gauge (analogous to Coulomb gauge in EM) and slowly changing so that second time derivatives go to zero, then take the trace of the spatial part. Note that in the transverse gauge, $\partial_i s^{ij} = 0$ and $\partial_i w^i = 0$. Also note that $\nabla^2 = \delta^{ij}\partial_i\partial_j$ and $\Box = -\partial_0^2 + \nabla^2$.

$$G_{ij} = (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) - \frac{1}{2}\partial_0\partial_i w_j - \frac{1}{2}\partial_0\partial_j w_i + 2\delta_{ij}\partial_0^2\Psi - \Box s_{ij} = 8\pi G T_{ij}$$

$$\delta^{ij}G_{ij} = (3\nabla^2 - \nabla^2)(\Phi - \Psi) - \frac{1}{2}\partial_0\partial_i w^i - \frac{1}{2}\partial_0\partial_j w^j + 6\partial_0^2\Psi + \delta^{ij}\partial_0^2 s_{ij} - \delta^{ij}\nabla^2 s_{ij} = 8\pi G(3\bar{p}) \quad (\bar{p} = \frac{1}{3}\delta^{ij}T_{ij})$$

$$\delta^{ij}G_{ij} = 2\nabla^2(\Phi - \Psi) + 6\partial_0^2\Psi + \delta^{ij}\partial_0^2 s_{ij} = 8\pi G(3\bar{p}) \qquad \text{(transverse gauge)}$$ [10]

$$\delta^{ij}G_{ij} = 2\nabla^2(\Phi - \Psi) = 8\pi G(3\bar{p}) \qquad \text{(slowly changing)}$$

[8] I copied the linearized Einstein's equations from the book, so they are probably missing some factors of c.
[9] The factor of 4 is also mentioned in Mach's Principle From Newton's Bucket to Quantum Gravity page 387
[10] The strain term drops because $\nabla^2 s_{ij} = (\delta_{kl}\partial^k\partial^l)s_{ij} = \delta_{kl}\partial^k(\partial^l s_{ij}) = 0$

$$G_{00} = 2\nabla^2\Psi = 8\pi G\rho \quad \Rightarrow \quad \nabla^2\Phi = 4\pi G(\rho + 3\bar{p})$$

Now it is interesting to note that if $\bar{p} = 0$, this equation for Φ becomes the same as the equation for Ψ. So if we add $\bar{p} = 0$ to our list of requirements, gravity will satisfy the exact same equations as electromagnetism.

Gravity obeys the same equations as electromagnetism up to multiplicative constants and signs under the approximations

(a) No feedback (linearized equations)

(b) Slowly changing (second order time derivatives drop)

(c) Traceless pressure ($\delta^{ij}T_{ij} = 0$)

From the obvious differences between the two forces, we had expected only the first approximation of no feedback would be necessary. In the future, we would like to understand how the last two requirements arise from the conceptual differences between the two forces. Ignazio Ciufolini suggests that the reason is the equivalence principle: "...general relativity, even the linearized theory, and electromagnetism are fundamentally different. Of course the main difference is the equivalence principle..."[11] He goes on to say that this difference stems from the fact that the ratio of gravitational mass to inertial mass is the same for all particles, but the ratio of charge to mass is not. However, I do not understand how this is relevant. Doctor D'Hoker proposed that the reason is that the graviton is a spin 2 particle, which causes different types of interaction. He also says that the fact that the graviton is spin 2 is responsible for the fact that gravity has no repulsion.[12] It would also be nice to have a complete characterization of when the two forces look the same since this list is sufficient, but not necessary.

9 Applications and Phenomena

A torus constructed as a ring of rings undergoing smoke-ring rotation can be used to accelerate a spaceship without applying any G-forces. An observer outside a spinning black hole will be dragged along with its rotation (frame dragging) and also torqued due to the difference in gravitomagnetic force being applied to either side of their body.

10 Gravitational Radiation

Gravitational radiation is so weak for two reasons, the obvious one being that Newton's constant is so small. The other is that the leading term comes from the quadrupole moment because a gravitational dipole oscillation would correspond to an oscillation of the center of mass, which is impossible. [13]

[11] Mach's Principle From Newton's Bucket to Quantum Gravity Page 390. Also in Gravitation and Inertia Page 353
[12] Question: Is it possible that some consequences of the nonlinearity snuck into the linearized equations?
[13] From Carroll page 305