Derivation of Maxwell's Equations

Chris Clark June 24, 2009

1 Introduction

Our goal in this paper is to derive Maxwell's equations in free space from basic quantum mechanics. In an earlier paper[1] we showed that the generator of time-evolution for any quantum state is an operator for a conserved scalar quantity. Usually the conserved scalar quantity is the energy, so the generator of time-evolution is proportional to the Hamiltonian. After inserting constants to fix dimensions, we obtain Schrödinger's equation

$$\hat{H}\Psi = i\hbar\frac{\partial}{\partial t}\Psi$$

Now let's apply Schrödinger's equation to a wave function for a photon. The state will have to be a vector since the electromagnetic field is vectorial. Special relativity requires that the operator for the energy of a photon is $\hat{E} = c|\hat{\mathbf{p}}|$. But this is not the operator that we will insert in Schrödinger's equation because there is another scalar conserved quantity for photons – helicity. The helicity operator is $\hat{\Lambda} = \hat{\mathbf{s}} \cdot (\hat{\mathbf{p}}/|\hat{\mathbf{p}}|)$, where $\hat{\mathbf{s}} = \hbar\hat{\boldsymbol{\sigma}}$, the vector of spin-1 Pauli matrices times $\hbar$. Since we have two scalar conserved quantities for photons, we construct the generator of time translation by multiplying these operators together.[2] We are free to multiply by scalar constants since constants do not change the fact that it is a conserved scalar quantity, so we divide by $\hbar$ to obtain units of energy. Therefore we have

$$\hat{H} = \hat{E}\hat{\Lambda}/\hbar = c|\hat{\mathbf{p}}|\hat{\mathbf{s}} \cdot (\hat{\mathbf{p}}/|\hat{\mathbf{p}}|)/\hbar = c(\hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{p}})$$

The spin-1 Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore

$$\hat{H} = c(\hat{\boldsymbol{\sigma}} \cdot \hat{p}) = c\begin{pmatrix} 0 & -i\hat{p}_z & i\hat{p}_y \\ i\hat{p}_z & 0 & -i\hat{p}_x \\ -i\hat{p}_y & i\hat{p}_x & 0 \end{pmatrix} = ic\widehat{\hat{\mathbf{p}}}$$

Here we have used the hat operator notation (the wide hat), which converts a vector into the matrix needed to execute the cross product operation by that vector i.e. if $\mathbf{u}$ and $\mathbf{v}$ are vectors, then $\widehat{\mathbf{u}}\mathbf{v} = \mathbf{u} \times \mathbf{v}$.

2 Curl Equations

Using $\hat{H} = ic\widehat{\hat{\mathbf{p}}}$, Schrödinger's equation gives

$$i\hbar\frac{\partial}{\partial t}\mathbf{\Psi} = \hat{H}\mathbf{\Psi} = ic\widehat{\hat{\mathbf{p}}}\mathbf{\Psi} = ic\hat{\mathbf{p}} \times \mathbf{\Psi}$$

The momentum operator is $\hat{\mathbf{p}} = -i\hbar\nabla$ so

$$i\hbar\frac{\partial}{\partial t}\mathbf{\Psi} = ic(-i\hbar)\nabla \times \mathbf{\Psi}$$

$$\nabla \times \mathbf{\Psi} = \frac{i}{c}\frac{\partial}{\partial t}\mathbf{\Psi}$$

[1] `http://dfcd.net/articles/fieldtheory/schrodinger.pdf`

[2] I don't know if there is a general rule that you are supposed to multiply all operators for conserved scalar quantities.

Now to get to Maxwell's equations, we need to relate $\mathbf{\Psi}$ to the electromagnetic fields. Of course all of the information about the fields has to be contained within the state, but it isn't immediately clear what the exact relationship is. However, we can use the fact that the magnitude squared of the state represents the probability density, which should be proportional to the electromagnetic field energy, $\frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2)$. Multiplying the field energy by $2\mu_0$, which has no effect on the resulting equations because both sides are linear in $\mathbf{\Psi}$, gives

$$\mathbf{\Psi}^*\mathbf{\Psi} = Re(\mathbf{\Psi})^2 + Im(\mathbf{\Psi})^2 \propto \frac{1}{c^2}E^2 + B^2$$

If we make the simplest assumption by setting $Re(\mathbf{\Psi}) = \frac{\mathbf{E}}{c}$ and $Im(\mathbf{\Psi}) = \mathbf{B}$, then we obtain the Riemann-Silberstein vector[3]

$$\mathbf{\Psi} = \frac{\mathbf{E}}{c} + i\mathbf{B}$$

Plugging into the previous curl equation, $\nabla \times \mathbf{\Psi} = \frac{i}{c}\frac{\partial}{\partial t}\mathbf{\Psi}$, we find

$$\nabla \times \left(\frac{\mathbf{E}}{c} + i\mathbf{B}\right) = \frac{i}{c}\frac{\partial}{\partial t}\left(\frac{\mathbf{E}}{c} + i\mathbf{B}\right)$$

This splits into two equations based on the real and imaginary parts

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

These are the two curl equations of Maxwell's equations in free space.

3 Divergence Equations

If we square $\hat{H}$ we get

$$\hat{H}^2 = \hat{E}^2\hat{\Lambda}^2/\hbar^2 = \hat{E}^2(\hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{p}}/|\hat{\mathbf{p}}|)^2 = \hat{E}^2(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}/|\hat{\mathbf{p}}|^2 + i\hat{\boldsymbol{\sigma}} \cdot (\hat{\mathbf{p}} \times \hat{\mathbf{p}})/|\hat{\mathbf{p}}|^2) = \hat{E}^2$$

where we used the identity $(\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma}) = (\mathbf{a} \cdot \mathbf{b})I + i\boldsymbol{\sigma}(\mathbf{a} \times \mathbf{b})$. Therefore we have

$$\hat{H}^2\mathbf{\Psi} = \hat{E}^2\mathbf{\Psi}$$

$$(ic\widehat{\mathbf{p}})^2\mathbf{\Psi} = \hat{c}^2\hat{p}^2\mathbf{\Psi}$$

$$-c^2\hat{\mathbf{p}} \times (\hat{\mathbf{p}} \times \mathbf{\Psi}) = c^2(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}})\mathbf{\Psi}$$

$$(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}})\mathbf{\Psi} - \hat{\mathbf{p}}(\hat{\mathbf{p}} \cdot \mathbf{\Psi}) = (\hat{\mathbf{p}} \cdot \hat{\mathbf{p}})\mathbf{\Psi}$$

$$i\hbar\nabla(\hat{\mathbf{p}} \cdot \mathbf{\Psi}) = 0$$

$$\hat{\mathbf{p}} \cdot \mathbf{\Psi} = const = 0$$

where we used the triple-product identity $\mathbf{a} \times (\mathbf{b} \times vc) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$. In the last line, the constant must be zero because any other value would be completely arbitrary. This is the transversality condition. Substituting the Riemann-Silberstein vector we obtain

$$\nabla \cdot \left(\frac{\mathbf{E}}{c} + i\mathbf{B}\right) = 0$$

Again, this splits into two equations based on the real and imaginary parts.

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

These are the divergence equations of Maxwell's equations in free space.

[3] The Riemann-Silberstein vector is often written as $\mathbf{F}$.

4 Conclusion

We showed how to derive Maxwell's equations in free space using basic quantum mechanics. Note that the main difference between this derivation and that for massive particles is the inclusion of a helicity factor in the generator of time-evolution. This suggests that the fact that helicity is preserved for massless particles has a significant impact on the resulting field equations. It would be interesting to understand better why that is the case. It would also be nice to understand the reason behind the form of the Riemann-Silberstein vector.

5 Reference

- F. Tamburini and D. Vicino, Photon wave function: A covariant formulation and equivalence with QED, Phys. Rev. A, Vol. 78, pp. 052116(1-5), 2008
- Raymer et al., The Maxwell wave function of the photon, p. 8, `http://www.uoregon.edu/~oco/Group_Pages/Raymer/Tutorials/TTRL5_V1.pdf`