Work-Energy Theorem

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1 Fundamental Equations

Definition of work:

$$W = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F}_{net} \cdot d\mathbf{r}$$

Definition of potential energy:

$$W_c = -\Delta U$$

Definition of Total Energy:

$$E = K + U$$

Segregation of conservative work:

$$W = W_c + W_{nc}$$

Work-energy theorem:

$$W = \Delta K$$

Non-conservation of Energy:

$$W_{nc} = \Delta E$$

2 The Work-Energy Theorem

Theorem 2.1. *If $\mathbf{r}(t)$ describes a smooth path, then $d\mathbf{r} = \mathbf{v}dt$.*

Proof. The quantity $d\mathbf{r}$ is the difference vector between two positions on the path that are infinitesimally close, so $d\mathbf{r} = \mathbf{r}(t + dt) - \mathbf{r}(t)$. But since the difference is infinitesimal, we can assume that the path is straight and the object is travelling with constant velocity. This is because if you zoom in far enough on a smooth curve, it will always look flat at every point and acceleration only increases velocity after a finite time has elapsed. So we can use the kinematic equation for straight-line motion at constant velocity, $\mathbf{x}_f = \mathbf{x}_i + \mathbf{v}t$. In this case, $\mathbf{r}(t + dt)$ is the final position, $\mathbf{r}(t)$ is the initial position, and dt is the time duration, so $\mathbf{r}(t + dt) = \mathbf{r}(t) + \mathbf{v}dt$. Therefore, $d\mathbf{r} = \mathbf{r}(t + dt) - \mathbf{r}(t) = \mathbf{v}dt$. □

Theorem 2.2. *For objects of constant mass, the total work done on the object during a process is equal to the change in kinetic energy of the object during that process, so $W = \Delta K$.*

Proof. We start with the definition of work.

$$W = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F}_{net} \cdot d\mathbf{r}$$

Since the mass of the object is constant, $\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt}$.

$$W = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \left(m\frac{d\mathbf{v}}{dt} \right) \cdot d\mathbf{r}$$

Now we perform a change of variables from $\mathbf{r}(t)$, which describes a trajectory as a function of time, to t using the relation $d\mathbf{r} = \mathbf{v}dt$. We must also change the limits of integration to the corresponding times.

$$W = m \int_{t_i}^{t_f} \frac{d\mathbf{v}}{dt} \cdot \mathbf{v}dt$$

By the product rule we have $\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} + \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = 2\frac{d\mathbf{v}}{dt} \cdot \mathbf{v}$, and dividing by 2 on each side we get $\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = \frac{1}{2}\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v})$, which is the integrand.

$$W = m \int_{t_i}^{t_f} \frac{1}{2}\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v})dt$$

By the second fundamental theorem of calculus,

$$W = \frac{1}{2}m(\mathbf{v} \cdot \mathbf{v})\Big|_{t=t_i}^{t=t_f}$$

$$W = \frac{1}{2}m(v_f^2 - v_i^2) = K_f - K_i = \Delta K$$

□