The Carnot Heat Engine

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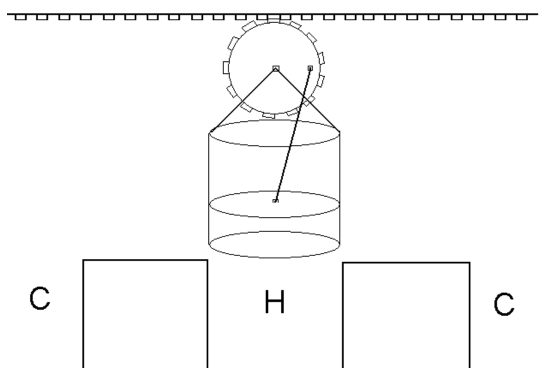
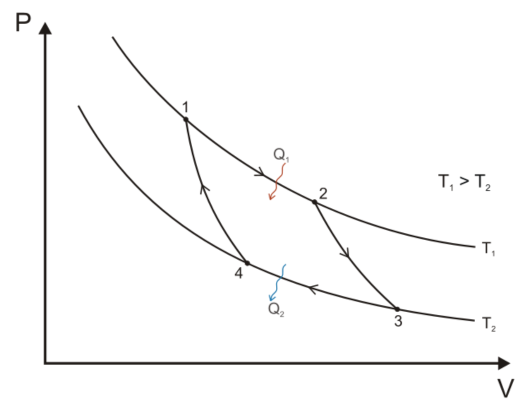
The Carnot heat engine is a theoretical device that produces mechanical work by pumping heat from a hot reservoir to a cold reservoir.[1] In a Carnot heat engine, a gas in a closed container with a movable piston goes through a cycle of expansion and compression known as the Carnot cycle. The stages of the Carnot cycle are: isothermal expansion, adiabatic expansion, isothermal compression, and adiabatic compression.[2] During isothermal expansion the gas is placed in thermal contact with a hot reservoir and during isothermal compresion the gas is placed in thermal contact with a cold reservoir. During the adiabatic processes, the gas is thermally insulated so that it cannot exchange heat with its surroundings. The adiabatic processes last just long enough so that the gas reaches the precise temperature of the next reservoir that it will come into thermal contact with. This is the complete description of a Carnot engine, so any engine that satisfies these properties is called a Carnot engine. However, no real-world engine could fully satisfy all these properties, so the Carnot heat engine is an idealized engine. Furthermore, the Carnot heat engine does not have any specific physical form; it is an abstract engine. However, to make things more concrete, we can choose one simple example of a Carnot heat engine, as shown in the following diagram.





In this diagram, the gas is stored in a cylindrical tank that is free to slide on a track above an infinite sequence of alternating hot and cold reservoirs with insulating barriers between them. The expansion and compression of the gas is mechanically linked to the motion of a massive gear that forces the tank-gear apparatus to crawl along the strip at the top. We assume that the whole system is frictionless and that everything besides the gear is massless.

Work is extracted whenever the piston moves in the same direction as the pressure of the gas, according to the equation $W = \int P \, dV$. So when the piston moves upward, work is extracted from the engine and when the piston moves downward, some of the previously extracted work is lost.

Stage 1. The pump starts on top of a hot reservoir, and the pressure of the gas pushes the piston upward, producing mechanical work. We assume that engine operates very slowly so that the gas in the pump remains in thermal equillibrium with the hot reservoir, so this is an isothermal expansion. As the piston is pushed upward, the gear assembly causes the pump to slide onto the thermally insulated region to the right.

Stage 2. The piston continues upwards from the pressure of the gas, producing more mechanical work. At this point, the thermal insulation prevents heat transfer, so this is an adiabatic expansion. During this adiabatic expansion, the temperature of the gas falls to the temperature of the cold reservoir. The upward motion of the piston keeps the gear spinning and the pump slides onto the cold reservoir to the right.

Stage 3. Now the momentum of the gear is transferred into a downward force on the piston. This process does work on the gas, which cancels out some of the work produced earlier. Since we are assuming

[1]The important point is that one is at a higher temperature than the other.

[2]Isothermal means the temperature is constant and adiabatic means that no heat flows into or out of the gas.

that the process is very slow, the gas in the pump remains in thermal equillibrium with the cold reservoir, so this is an isothermal compression. The momentum of the gear causes the pump to slide onto the thermally insulated region to the right.

Stage 4. The gear still has some momentum left, which causes the piston to be forced further downward. This does additional work on the gas, which cancels out yet more of the work produced earlier. Since the gas is thermally insulated, this is an adiabatic compression. During this adiabatic compression, the temperature of the gas rises to the temperature of the hot reservoir. The momentum of the gear causes the pump to slide onto the hot reservoir to the right and the cycle repeats.

In stages 1 and 2 energy is extracted from the gas, but in stages 3 and 4 some energy is put back into the gas. We need to determine whether the net extracted energy is positive to see if the engine has the ability to do sustained mechanical work. The best way to see this is by looking at a pressure vs. volume diagram because the work that the gas does is given by $W = \int P\ dV$, which is the area under the curve on the pressure vs. volume diagram. First we need to find the pressure-volume relations for isothermal and adiabatic processes.

During isothermal processes in which T is constant, the ideal gas law indicates that $PV = \text{constant}$. Therefore, $P \propto 1/V$. Every temperature has a corresponding isotherm curve of this form on the P-V diagram.

During adibatic processes in which $Q = 0$, the differential form of the first law of thermodynamics $dE_{int} = dQ + dW$ becomes $dE_{int} = dW$. Or since $dE_{int} = nC_V dT$ and $dW = -pdV$, this is equivalent to $nC_V dT = -PdV$. Staring from the differential form of the ideal gas law,

$$d(PV) = PdV + VdP = nRdT$$

$$PdV + VdP = -(R/C_V)PdV$$

$$(1 + R/C_V)PdV + VdP = 0$$

$$\frac{C_V + R}{C_V}\frac{dV}{V} + \frac{dP}{P} = 0$$

$$\gamma\frac{dV}{V} + \frac{dP}{P} = 0$$

Integrating,

$$\gamma \ln(V) + \ln(P) = \text{constant}$$

Exponentiating,

$$PV^\gamma = \text{constant}$$

With these results and the fact that $\gamma > 1$, the general shape of the P-V diagram shown above is determined.[3] The net work done by the gas is then the area under the expansion phases minus the area under the compression phases. This corresponds to the positive value of the area enclosed by the curve of the complete cycle, which means that the engine does positive net work. The trick that makes this all work is that the expansion phase occurs at a higher average temperature, which produces a higher average pressure than during the compression phase. Higher pressures do more work over the same volume change, so it is possible to recompress the gas without doing as much work as long as the temperature is lower during compression.

[3]Image courtesy of Keta (Wikipedia).