# Radial Force Deflection


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## INTRODUCTION

Why is it that a figure skater spins faster when they pull their arms inwards? Conservation of angular momentum gives us a quick answer. Take the simplifying case of a point mass at radius r and rotational speed v.

$$L = I\omega = mr^2 v/r = mrv$$

$$\frac{dL}{dr} = 0 \Rightarrow mv + mr\frac{dv}{dr} = 0 \Rightarrow \frac{dv}{dr} = -\frac{v}{r} \tag{1}$$

However, this does not really shed any light on the mechanism that causes the acceleration. Any acceleration is caused by a force and it may not be obvious how that force operates in this situation. By using conservation of angular momentum the mechanism is abstracted away. This paper explains the mechanism, which we call radial force deflection, directly in terms of Newton's laws.

## EXPLANATION

Naturally our starting point is Newton's second law

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}$$

We are concerned with changes in speed, so first we find the time derivative of $p = |\mathbf{p}|$.

$$\frac{dp}{dt} = \frac{d}{dt}\left(\frac{\mathbf{p}}{\hat{\mathbf{p}}}\right) = \frac{\hat{\mathbf{p}} \cdot \dot{\mathbf{p}} - \mathbf{p} \cdot \dot{\hat{\mathbf{p}}}}{\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}} = \mathbf{F} \cdot \hat{\mathbf{p}}$$

where the second term is zero because a unit vector can never change in the direction of itself since it's length is constant.

In Fig. 1, the point mass is spiraling inwards for some duration. During its inward spiral, it has a component of momentum in the radial direction and that means $\mathbf{F} \cdot \hat{\mathbf{p}} \neq 0$, so the speed increases. Thus the explanation is that the radial force causes an acceleration of the point mass during its inward spiral due to the fact that its momentum has a component in the radial direction.

## CALCULATION

Now we must check that this explanation implies equation (1) for the rate of change of velocity with respect to radius. Our strategy is to look at the inward spiral for an infinitesimal time interval $dt$. During this interval the spiral path will form a constant angle with the circular path at the same radius, call this angle $\phi$. Then we have

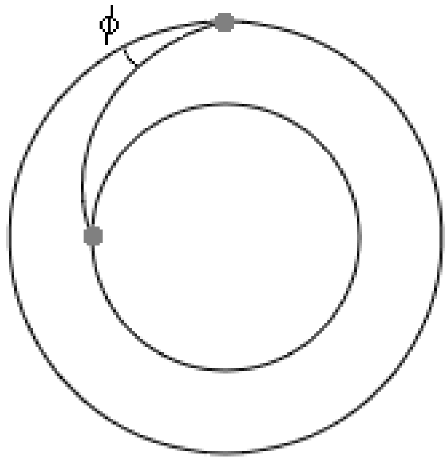


FIG. 1: Spiraling inward

$$\frac{dp}{dt} = \mathbf{F} \cdot \hat{\mathbf{p}} = F\sin(\phi) \Rightarrow \frac{dv}{dt} = a\sin(\phi) \tag{2}$$

The next step is to find an expression for $\phi$. In Fig. 2 we see that if we zoom in on the spiral path so that it only shows the time interval $dt$, then it becomes a straight line because it must intersect the two arcs with the same angle and the arcs are straight lines. Now define the radial speed

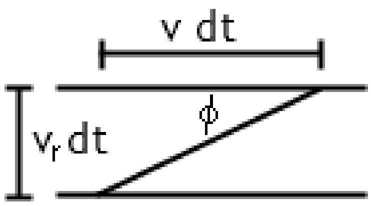


FIG. 2: Zooming in

$$v_r = \left|\frac{dr}{dt}\right| \tag{3}$$

Then [1]

$$\tan(\phi) = \frac{v_r dt}{v dt} = \frac{v_r}{v} \tag{4}$$

We can now find $\sin(\phi)$ using trigonometric identities.

$$\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)} \quad and \quad \cos^2(\phi) = 1 - \sin^2(\phi)$$

$$\sin^2(\phi) = \tan^2(\phi)(1 - \sin^2(\phi))$$

$$\sin^2(\phi)(1 + \tan^2(\phi)) = \tan^2(\phi)$$

$$\sin(\phi) = \frac{\tan(\phi)}{\sqrt{1 + \tan^2(\phi))}} = \frac{v_r/v}{\sqrt{1 + v_r^2/v^2}}$$

$$\sin(\phi) = \frac{v_r}{\sqrt{v^2 + v_r^2}} \tag{5}$$

So by equation (2),

$$\frac{dv}{dt} = \frac{a v_r}{\sqrt{v^2 + v_r^2}} \tag{6}$$

Now we will make an assumption. We assume that $v$ is a function only of $r$ and the initial conditions, which means that the velocity at a particular radius does not depend on how the radial force was applied. This allows us to write

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = -\frac{dv}{dr} v_r \tag{7}$$

by the chain rule. Here we have taken the case that the point mass is being pulled inwards, hence the negative sign. Inserting equation (6),

$$\frac{dv}{dr} = -\frac{a}{\sqrt{v^2 + v_r^2}} \tag{8}$$

The final step is to find an expression for $v_r$ in terms of $a$ (which is proportional to the applied force). This is done by generalizing the textbook derivation of the centripetal force to cases where the initial and final velocities are not the same.

Fig. 3 depicts the change in the velocity during an infinitesimal time interval. The first thing to notice is that the angle $\alpha$ is still infinitesimal despite the fact that the velocity vector is rotated by the angle $\phi$ from the tangential direction because the rotation by $\phi$ is present in both vectors. The angle $\alpha$ is thus purely due to the rotation of the point mass at radius $r$,

$$\alpha = \omega dt = \frac{v}{r} dt \tag{9}$$

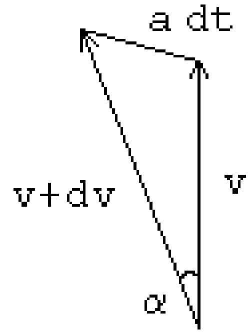


FIG. 3: Velocity vectors

To get the desired expression for $v_r$, we start with the law of cosines and keep terms of order $dt^2$.

$$c^2 = a^2 + b^2 - 2ab\cos(\theta_c)$$

$$(adt)^2 = v^2 + (v + dv)^2 - 2v(v + dv)\cos(\alpha)$$

$$a^2 dt^2 = 2v^2 + 2vdv + dv^2 - (2v^2 + 2vdv)\cos(\alpha)$$

$$a^2 dt^2 = (2v^2 + 2vdv)(1 - \cos(\alpha)) + dv^2$$

$$a^2 dt^2 = (2v^2 + 2vdv)(1 - (1 - \alpha^2/2)) + dv^2$$

$$a^2 dt^2 = (2v^2 + 2vdv)\left(\frac{1}{2}\frac{v^2}{r^2} dt^2\right) + dv^2$$

Now by equation (2), $dv = a\sin(\phi)dt$ so the $2vdv$ term is of order $dt^3$ and we drop it.

$$a^2 dt^2 = \frac{v^4}{r^2} dt^2 + a^2 \sin^2(\phi) dt^2$$

$$a^2 - \frac{v^4}{r^2} = a^2 \sin^2(\phi)$$

The second term on the left is the square of the well-known centripetal acceleration $a_c$. [2] Inserting equation (5) for $\sin(\phi)$,

$$a^2 - a_c^2 = a^2 \frac{v_r^2}{v^2 + v_r^2}$$

$$(a^2 - a_c^2)(v^2 + v_r^2) = a^2 v_r^2$$

$$a^2 v^2 - a_c^2 v^2 + a^2 v_r^2 - a_c^2 v_r^2 = a^2 v_r^2$$

$$a^2v^2 - a_c^2v^2 - a_c^2v_r^2 = 0$$

$$v_r^2 = \frac{v^2}{a_c^2}(a^2 - a_c^2)$$

$$v_r = \frac{v}{a_c}\sqrt{a^2 - a_c^2}$$

$$v_r = \frac{r}{v}\sqrt{a^2 - a_c^2} \tag{10}$$

Finally we plug equation (10) into equation (8).

$$\frac{dv}{dr} = -\frac{a}{\sqrt{v^2 + \frac{r^2}{v^2}(a^2 - a_c^2)}} = -\frac{a}{ra/v}$$

$$\frac{dv}{dr} = -\frac{v}{r} \tag{11}$$

Now it is clear that our explanation gives the same result as using conservation of angular momentum as seen in equation (1).

## CONCLUSION

We have found in two pages what angular momentum tells us in two lines, but we have gained an insight into rotational motion that angular momentum overlooks.

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[1] You may think that $vdt$ would be along the diagonal path, but $vdt$ is how far the mass would travel in a time $dt$ if it was at the constant velocity v. The diagonal path is a result of an acceleration and so is longer than $vdt$.

[2] We could have skipped to this point by starting with $a = \sqrt{a_c^2 + (dv/dt)^2}$, but this equation is certainly not obvious.