Complex Partial Derivatives

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Let f be a function of two independent complex parameters z_1 and z_2. We can reparameterize f to express it in terms of just one complex variable and its complex conjugate by defining $z = z_1 + iz_2$ and $z^* = z_1 - iz_2$. This allows us to write $f(z_1, z_2) = f(z, z^*)$. The partial derivative of f with respect to z holds z^* constant on the approach to the limit.

$$0 = \Delta z^* = (z_1 + \Delta z_1) - i(z_2 + \Delta z_2) - (z_1 - iz_2) = \Delta z_1 - i\Delta z_2$$

Therefore, the partial derivative is

$$\begin{aligned}
\frac{\partial f(z, z^*)}{\partial z} &= \lim_{\substack{\Delta z \to 0 \\ \Delta z^* = 0}} \frac{f(z + \Delta z, z^*) - f(z, z^*)}{\Delta z} \\
&= \lim_{\substack{\Delta z_1, z_2 \to 0 \\ \Delta z_1 - i\Delta z_2 = 0}} \frac{f(z_1 + \Delta z_1, z_2 + \Delta z_2) - f(z_1, z_2)}{\Delta z_1 + i\Delta z_2} \\
&= \lim_{\substack{\Delta z_1, z_2 \to 0 \\ \Delta z_1 - i\Delta z_2 = 0}} \frac{f(z_1 + \Delta z_1, z_2 + \Delta z_2) - f(z_1, z_2 + \Delta z_2) + f(z_1, z_2 + \Delta z_2) - f(z_1, z_2)}{\Delta z_1 + i\Delta z_2} \\
&= \lim_{\substack{\Delta z_1, z_2 \to 0 \\ \Delta z_1 - i\Delta z_2 = 0}} \frac{f(z_1 + \Delta z_1, z_2 + \Delta z_2) - f(z_1, z_2 + \Delta z_2)}{\Delta z_1 + i\Delta z_2} + \lim_{\substack{\Delta z_1, z_2 \to 0 \\ \Delta z_1 - i\Delta z_2 = 0}} \frac{f(z_1, z_2 + \Delta z_2) - f(z_1, z_2)}{\Delta z_1 + i\Delta z_2} \\
&= \lim_{\Delta z_1 \to 0} \frac{f(z_1 + \Delta z_1, z_2 - i\Delta z_1) - f(z_1, z_2 - i\Delta z_1)}{\Delta z_1 + \Delta z_1} + \lim_{\Delta z_2 \to 0} \frac{f(z_1, z_2 + \Delta z_2) - f(z_1, z_2)}{i\Delta z_2 + i\Delta z_2} \\
&= \frac{1}{2}\frac{\partial f}{\partial z_1} + \frac{1}{2i}\frac{\partial f}{\partial z_2}
\end{aligned}$$

If we now restrict z_1 and z_2 to be real, they become the real and imaginary parts of z, which we write as x and y respectively. Therefore, if $z = x + iy$ for $x, y \in \mathbb{R}$,

$$\frac{\partial f(z, z^*)}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y}$$

Similarly,

$$\frac{\partial f(z, z^*)}{\partial z^*} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y}$$

These partial derivatives can be evaluated by treating z and z^* as if they were completely independent parameters, which explains the reason for treating ϕ and ϕ^* as independent fields in quantum field theory.