# Isospin and Noether's Theorem

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We will derive the law of conservation of the third component of isospin ($I_3$) from the Lagrangian for free nucleons using Noether's theorem. The Lagrangian density for free nucleons is based on the Dirac equation as follows, [1]

$$\mathcal{L} = \bar{\psi}_p(i\gamma^\mu\partial_\mu - m)\psi_p + \bar{\psi}_n(i\gamma^\mu\partial_\mu - m)\psi_n$$

where $\bar{\psi} \equiv \psi^\dagger\gamma^0$. Recall that the $\psi_p$ and $\psi_n$ are 4-spinors, the first referring to protons and the second referring to neutrons. In order to make the notation more compact, we can define a column vector of the two 4-spinors

$$\psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}$$

and we notice that $\mathcal{L}$ can now be written as

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

because this is

$$(\psi_p^\dagger \; \psi_n^\dagger)\gamma^0(i\gamma^\mu\partial_\mu - m)\begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}$$

$$= (\bar{\psi}_p \; \bar{\psi}_n)(i\gamma^\mu\partial_\mu - m)\begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}$$

$$= (\bar{\psi}_p \; \bar{\psi}_n)\begin{pmatrix} (i\gamma^\mu\partial_\mu - m)\psi_p \\ (i\gamma^\mu\partial_\mu - m)\psi_n \end{pmatrix}$$

$$= \bar{\psi}_p(i\gamma^\mu\partial_\mu - m)\psi_p + \bar{\psi}_n(i\gamma^\mu\partial_\mu - m)\psi_n = \mathcal{L}$$

Because the proton and neutron wave-functions show up in the same way in this Lagrangian, exchanging part of the proton field with part of the neutron field will not change the value of the Lagrangian. To get the strongest constraint, we want to consider the most general form of this invariance, which turns out to be application of matrices in SU(2) to $\psi$.[2] So changing $\psi$ according to

$$\psi' = U\psi \quad (U \in SU(2))$$

leaves the Lagrangian invariant. We say that $\mathcal{L}$ has an $SU(2)$ symmetry, which is isospin symmetry in this case.

In order to turn this into a conservation law for $I_3$ we need to use Noether's theorem. We will now provide a proof of a restricted form of Noether's theorem.[3] Say that a field $\phi$ is transformed infinitesimally,

$$\phi'(x) = \phi(x) + d\phi(x)$$

and assume this transformation does not affect the form of the Lagrangian so that we can rightly call the transformation a symmetry.[4] So we know that the change in the Lagrangian evaluated using the chain rule will give zero,

[1] Question: Do we need to worry about the potential due to the strong interaction?

[2] Todo: Show this!

[3] Adapted from Peskin and Schroeder page 17.

[4] It can still be a symmetry if the Lagrangian changes by only a gradient since that would not change the equations of motion that result from the Euler-Lagrange equations.

$$
\begin{aligned}
0 = d\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi} d\phi + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) d(\partial_\mu \phi) \\
&= \frac{\partial \mathcal{L}}{\partial \phi} d\phi + \left[-\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) d\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) d\phi\right] + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) \partial_\mu (d\phi) \\
&= \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\right] d\phi + \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) d\phi + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) \partial_\mu (d\phi)\right]
\end{aligned}
$$

The first square bracket is zero by the Euler-Lagrange equation and the second square bracket is just a product rule. Therefore we have the conserved current equation

$$
\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} d\phi\right) = 0
$$

We can generalize this result slightly to the case where multiple fields are being transformed simultaneously. We just create a new index $i$ that runs over all fields that are being transformed so

$$
\phi_i'(x) = \phi_i(x) + d\phi_i(x)
$$

and the derivation carries through exactly the same with the resulting conserved current expression containing the sum of conserved currents for each field,

$$
\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} d\phi_i\right) = 0
$$

To get a conserved charge, we sacrifice some generality because the local conservation of currents becomes a mere global conservation of charge. But the sacrifice is justified by the increase in conceptual simplicity. The expression for conserved charge comes from integrating the current conservation equation over all space

$$
\begin{aligned}
0 &= \int \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} d\phi_i\right) \, d^3x \\
&= \int -\frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_i)} d\phi_i\right) + \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial(\nabla \phi_i)} d\phi_i\right) \, d^3x
\end{aligned}
$$

The last term is zero by the divergence theorem and the fact that the Lagrangian goes to zero at infinity. So we get

$$
0 = \frac{\partial}{\partial t} \int \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_i)} d\phi_i\right) \, d^3x
$$

which gives a natural definition for a conserved charge as

$$
Q \equiv \int \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_i)} d\phi_i\right) \, d^3x
$$

that is clearly constant in time.

Now we can ask what Noether's theorem tells us about isospin. First we need to check that our assumptions are satisfied, i.e. does the Lagrangian remain invariant under infinitesimal transformations of $\psi$?

$$
\begin{aligned}
\mathcal{L}' &= (U\psi)^\dagger \gamma^0 (i\gamma^\mu \partial_\mu - m) U\psi \\
&= \psi^\dagger U^\dagger \gamma^0 (i\gamma^\mu \partial_\mu - m) U\psi \\
&= \psi^\dagger \gamma^0 (i\gamma^\mu \partial_\mu - m) U^\dagger U\psi
\end{aligned}
$$

since $U$ commutes with the $\gamma^0$ because they are acting on different spaces.

$$= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi = \mathcal{L}$$

since $U$ is unitary because all matrices in $SU(2)$ are unitary. Therefore the Lagrangian does satisfy our assumptions and we can apply Noether's theorem. First we compute the derivative of the Lagrangian for $\psi_p$

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_p)} = \psi_p^\dagger \gamma^0 i \gamma^0 = i\psi_p^\dagger$$

Similarly the result for $\psi_n$ is $i\psi_n^\dagger$.

Now, if we want to apply an infinitesimal transformation from the $SU(2)$ group, then we want to look at the generators. Any element $U$ in $SU(2)$ can be expressed as

$$U = e^{\alpha_i \sigma_i}$$

where the $\sigma_i$ are the Pauli matrices and the $\alpha_i$ are constant coefficients. In this case, the Pauli matrices are the generators. An infinitesimal transformation is found by shrinking the coefficients to infinitesimal size and removing terms of second order or higher in infinitesimals,

$$U = I + d\alpha_i \sigma_i$$

where $d\alpha_i$ are infinitesimal coefficients.

For simplicity, we choose the diagonal generator, which is $\sigma_z$ and apply it with Noether's theorem. The only thing the transformation affects is $d\phi$, which is $d\psi$ in this case.

$$d\psi(x) = \psi'(x) - \psi(x) = (I + d\alpha\sigma_z)\psi - \psi$$

$$= d\alpha\sigma_z\psi = d\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} = d\alpha \begin{pmatrix} \psi_p \\ -\psi_n \end{pmatrix}$$

Therefore, the conserved charge is

$$Q \equiv \int \left( \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_i)} d\psi_i \right) \, d^3x$$

$$= \int \left( \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_p)} d\psi_p \right) + \left( \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_n)} d\psi_n \right) \, d^3x$$

$$= \int i\psi_p^\dagger \psi_p d\alpha - i\psi_n^\dagger \psi_n d\alpha \, d^3x$$

Dividing by the constant $id\alpha$ gives

$$Q = \int |\psi_p|^2 - |\psi_n|^2 \, d^3x$$

$$= N_p - N_n$$

where $N_p$ and $N_n$ are the number of protons and neutrons respectively. This conserved charge is actually the third component of isospin, so we have shown that the $SU(2)$ symmetry of the Lagrangian implies the conservation of $I_3 = N_p - N_n$ through Noether's theorem.

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