

Quantum Field Theory Notes

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1 Klein-Gordon Equation

We will begin by assuming the standard time-dependent Schrodinger equation.

$$H\phi = i\hbar \frac{\partial}{\partial t} \phi$$

Quantum Field Theory attempts to be a completely accurate theory of nature, so it must be compatible with relativity. In the effort to reconcile Schrodinger's equation with relativity, we will stumble upon the Klein-Gordon equation. Relativity states that the Hamiltonian for a free particle is $H = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4}$. In this expression, c is a universal constant, and m is a constant for the particle under consideration, but \mathbf{p} poses more of an issue because it is a differential operator, namely $\mathbf{p} = i\hbar \nabla$. This concerns us because we don't usually deal with a square root of a differential operator. If we insert this Hamiltonian into Schrodinger's equation, we get

$$\sqrt{\mathbf{p}^2 c^2 + m^2 c^4} \phi = i\hbar \frac{\partial}{\partial t} \phi$$

It is possible to fix our problem by squaring the equation, or more rigorously, we let $\phi \rightarrow H\phi$ so we obtain

$$(\mathbf{p}^2 c^2 + m^2 c^4) \phi = i\hbar \frac{\partial}{\partial t} H\phi = -\hbar^2 \frac{\partial^2}{\partial t^2} \phi$$

Now inserting $\mathbf{p} = -i\hbar \nabla$ and rearranging,

$$m^2 c^4 \phi = \left(c^2 \hbar^2 \nabla^2 - \hbar^2 \frac{\partial^2}{\partial t^2} \right) \phi$$

$$m^2 c^4 \phi = c^2 \hbar^2 \left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \phi$$

$$m^2 c^2 \phi = \hbar^2 \square^2 \phi$$

$$\square^2 \phi - \frac{m^2 c^2}{\hbar^2} \phi = 0$$

which is the Klein-Gordon equation. This equation assumes nothing about the properties (such as spin) of the particle it describes, so it should work for any particle. ¹

It seems like we have accomplished our goal rather quickly, and in fact it was too good to be true. The solutions to the Klein-Gordon equation *do not* provide the correct free particle probability distributions. We know this without ever having to do an experiment because it is theoretically flawed. The solutions permit negative probability densities, which are clearly unrealistic. ²

But our efforts are not all lost. We will see that the Klein-Gordon equation is crucially important as long as it is re-interpreted properly, which is as something we will call a quantized field equation. But first we need to do a little math to translate the equations of motion into a Lagrangian formalism. Now if we integrate the Klein-Gordon equation we will get zero since the integrand is always zero.

$$\int \left(\square^2 - \frac{m^2 c^2}{\hbar^2} \right) \phi \delta\phi \, d^3x \, dt = 0$$

$$\int \left(\nabla^2 \phi \delta\phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} \delta\phi - \frac{m^2 c^2}{\hbar^2} \phi \delta\phi \right) d^3x \, dt = 0$$

¹See Ryder page 29. Question: Does it work for massless particles?

²Question: Could it be that those solutions just need to be ruled out?

Integration by parts in the first and second terms gives

$$\int \left(-\frac{\partial\phi}{\partial x} \frac{\partial\delta\phi}{\partial x} - \frac{\partial\phi}{\partial y} \frac{\partial\delta\phi}{\partial y} - \frac{\partial\phi}{\partial z} \frac{\partial\delta\phi}{\partial z} + \frac{1}{c^2} \frac{\partial\phi}{\partial t} \frac{\partial\delta\phi}{\partial t} - \frac{m^2 c^2}{\hbar^2} \phi \delta\phi \right) d^3x dt = 0$$

$$\int \left(-\frac{1}{2} \delta \left(\left(\frac{\partial\phi}{\partial x} \right)^2 + \left(\frac{\partial\phi}{\partial y} \right)^2 + \left(\frac{\partial\phi}{\partial z} \right)^2 \right) + \frac{1}{2} \frac{1}{c^2} \delta \left(\left(\frac{\partial\phi}{\partial t} \right)^2 \right) - \frac{1}{2} \frac{m^2 c^2}{\hbar^2} \delta(\phi^2) \right) d^3x dt = 0$$

$$\delta \int \left(-\frac{1}{2} (\nabla\phi)^2 + \frac{1}{2} \frac{1}{c^2} \left(\frac{\partial\phi}{\partial t} \right)^2 - \frac{1}{2} \frac{m^2 c^2}{\hbar^2} \phi^2 \right) d^3x dt = 0$$

This is the equation for the action being stationary, therefore the Lagrangian density is

$$\mathcal{L} = \frac{1}{2} \left[\frac{1}{c^2} \dot{\phi}^2 - (\nabla\phi)^2 - \frac{m^2 c^2}{\hbar^2} \phi^2 \right]$$

Notice that the right hand side has units of inverse length squared if ϕ is dimensionless, whereas it should have units of energy per unit length cubed. It may be useful to assign ϕ the units of length, as if it were the coordinate representing the position of a mass on a spring. Then the right hand side is dimensionless. We can then give it the correct units by multiplying by ρc^2 where ρ is a mass density.

$$\mathcal{L} = \frac{1}{2} \rho \dot{\phi}^2 - \frac{1}{2} \rho c^2 (\nabla\phi)^2 - \frac{1}{2} \rho \frac{m^2 c^4}{\hbar^2} \phi^2$$

Note that m is the mass of the particle being described, whereas ρ is an independent mass density that is based on the properties of the vacuum. We can think of this equation using a mechanical analogy. We can find a mechanical system whose Lagrangian would take this form. It turns out to be a lattice of masses connected by springs with an additional spring connecting each mass to its equilibrium point. It is easier to view this lattice if we reduce its dimension by one to make it a two-dimensional mattress. Then there is a two-dimensional grid of springs connecting point masses and a spring underneath each point mass connecting it to a solid surface.

The first term in the Lagrangian corresponds to the kinetic energy of the point mass, the second term corresponds to the energy of the lateral springs, and the third term corresponds to the potential energy of the vertical springs. Notice the potential terms are negative as they should be according to the prescription $L = T - V$.

2 Where are the particles?

Carroll states on page 387, "In a Schrodinger-picture quantization of the field theory, we would define a complex wave functional $\Psi[\phi(x^\mu)]$, which would represent the probability amplitude for finding the field in each configuration." Wikipedia has an article called "Schrodinger Functional", which says

$$\Psi[\phi_2, t_2] = \int \mathcal{D}\phi_1 \left\langle \phi_2 \left| e^{-iH(t_2-t_1)/\hbar} \right| \phi_1 \right\rangle \Psi[\phi_1, t_1]$$

Ryder (4.31) is

$$\psi(x) = \left\langle 0 \left| \hat{\phi}(x) \right| p \right\rangle = e^{-ipx}$$

and there should be a $1/\hbar$ in the exponent.

Sakurai (1.7.32) is

$$\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

$$e^{-ipx/\hbar} = \sqrt{2\pi\hbar} \langle p | x \rangle$$

$$e^{ipx/\hbar} = \sqrt{2\pi\hbar} \langle p | -x \rangle$$

Applying the definition of Hermitian conjugate to Ryder's equation,

$$\begin{aligned}\langle p | \hat{\phi}^\dagger(x) | 0 \rangle &= e^{ipx/\hbar} \\ \langle p | \hat{\phi}^\dagger(x) | 0 \rangle &= \sqrt{2\pi\hbar} \langle p | -x \rangle\end{aligned}$$

So it may be the case that

$$\hat{\phi}^\dagger(x) | 0 \rangle = \sqrt{2\pi\hbar} | -x \rangle$$

So far everything looks really straightforward and it appears that we have a Lagrangian whose solutions will give us the wavefunction for a particle. However, this is all wrong because the Klein-Gordon equation solutions do not give true probability densities. According to Professor Zvi Bern, this whole motivation for the Lagrangian and the upcoming second quantization are historical accidents. As it turns out, the currently accepted motivation for quantum field theory is not quite so deductive; it requires assuming that the field is a continuous lattice of coupled quantum oscillators. Thanks to Professor Per Kraus, we finally discovered the conceptual foundation. The key is to start with this continuous lattice of coupled quantum oscillators and just apply normal quantum mechanics to it. If the field represents one particle, then that particle's wavefunction is given by the solution to Schrodinger's equation with the Klein-Gordon Hamiltonian.

$$H[\phi]\Psi[\phi] = E\Psi[\phi]$$

Thus our next step will be to find $H[\phi]$.

In classical mechanics we had the formula

$$H = \sum_k p_k \dot{q}_k - L$$

Now we are modeling a system with a continuously infinite number of degrees of freedom, which are the amplitudes of the oscillators at each point in space. So we must convert the sum to an integral and use $\phi(\mathbf{x})$ as our coordinates and $\pi(\mathbf{x})$ as our momenta.

$$\begin{aligned}H &= \int \pi(\mathbf{x}) \dot{\phi}(\mathbf{x}) d^3x - L \\ \int \mathcal{H} d^3x &= \int \pi(\mathbf{x}) \dot{\phi}(\mathbf{x}) - \int \mathcal{L} d^3x \\ \mathcal{H} &= \pi(\mathbf{x}) \dot{\phi}(\mathbf{x}) - \mathcal{L}\end{aligned}$$

This is not the final result because in the Hamiltonian formalism, there are no explicit time derivatives because they are replaced with the expression for momentum. In classical mechanics we had $p_k \equiv \frac{\partial L}{\partial \dot{q}_k}$. Now we have the functional derivative

$$\pi(\mathbf{x}) = \frac{\delta L}{\delta \dot{\phi}(\mathbf{x})}$$

Instead of solving this specifically, we can prove a general theorem:

$$\frac{\delta L}{\delta \phi} = \frac{\partial \mathcal{L}}{\partial \phi}$$

where $L[\phi(\mathbf{x})] = \int \mathcal{L}[\phi(\mathbf{x})] d^3x$. By the definition of functional differentiation,

$$\begin{aligned}\frac{\delta L[\phi(\mathbf{x})]}{\delta \phi(\mathbf{y})} &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int \mathcal{L}[\phi(\mathbf{x}) + \epsilon \delta(\mathbf{x} - \mathbf{y})] - \mathcal{L}[\phi(\mathbf{x})] d^3x \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int \mathcal{L}[\phi(\mathbf{x})] + \epsilon \delta(\mathbf{x} - \mathbf{y}) \frac{\partial \mathcal{L}[\phi(\mathbf{x})]}{\partial \phi(\mathbf{x})} + \mathcal{O}(\epsilon^2) - \mathcal{L}[\phi(\mathbf{x})] d^3x \\ &= \int \delta(\mathbf{x} - \mathbf{y}) \frac{\partial \mathcal{L}[\phi(\mathbf{x})]}{\partial \phi(\mathbf{x})} d^3x \\ &= \frac{\partial \mathcal{L}[\phi(\mathbf{y})]}{\partial \phi(\mathbf{y})} \quad 3\end{aligned}$$

³Are they both supposed to go to y?

So we now have

$$\pi(\mathbf{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\mathbf{x})} = \rho \dot{\phi}(\mathbf{x})$$

Therefore

$$\begin{aligned} \mathcal{H} &= \pi(\mathbf{x}) \frac{\pi(\mathbf{x})}{\rho} - \mathcal{L} \\ \mathcal{H} &= \frac{\pi^2(\mathbf{x})}{\rho} - \frac{\pi^2(\mathbf{x})}{2\rho} + \frac{1}{2} \rho c^2 (\nabla \phi)^2 + \frac{1}{2} \rho \frac{m^2 c^4}{\hbar^2} \phi^2(\mathbf{x}) \\ \mathcal{H} &= \frac{\pi^2(\mathbf{x})}{2\rho} + \frac{1}{2} \rho c^2 (\nabla \phi)^2 + \frac{1}{2} \rho \frac{m^2 c^4}{\hbar^2} \phi^2(\mathbf{x}) \end{aligned}$$

3 Quantum Mechanical Wavefunction

Aitchison page 136 explains that the single particle quantum mechanical wavefunction is given by

$$\psi(x, t) = \langle 0 | \hat{\phi}(x, t) | k' \rangle$$

where $\hat{\phi}(x, t)$ is given by (5.116)

$$\hat{\phi} = \int_{-\infty}^{\infty} \frac{dk}{2\pi\sqrt{2\omega}} \left[\hat{a}(k) e^{ikx - i\omega t} + \hat{a}^\dagger(k) e^{-ikx + i\omega t} \right]$$

and by (5.130)

$$|k'\rangle = N \hat{a}^\dagger(k') |0\rangle$$

Therefore

$$\begin{aligned} \psi(x, t) &= \langle 0 | \int_{-\infty}^{\infty} \frac{dk}{2\pi\sqrt{2\omega}} \left[\hat{a}(k) e^{ikx - i\omega t} + \hat{a}^\dagger(k) e^{-ikx + i\omega t} \right] N \hat{a}^\dagger(k') |0\rangle \\ &= \langle 0 | \int_{-\infty}^{\infty} \frac{dk}{2\pi\sqrt{2\omega}} \hat{a}(k) e^{ikx - i\omega t} N \hat{a}^\dagger(k') |0\rangle \end{aligned}$$

Now we use the commutation relation from (5.117), $[\hat{a}(k), \hat{a}^\dagger(k')] = 2\pi\delta(k - k')$

$$\begin{aligned} &= \langle 0 | \int_{-\infty}^{\infty} \frac{dk}{2\pi\sqrt{2\omega}} N e^{ikx - i\omega t} \left(2\pi\delta(k - k') + \hat{a}^\dagger(k') \hat{a}(k) \right) |0\rangle \\ &= \langle 0 | \frac{1}{\sqrt{2\omega'}} N e^{ik'x - i\omega't} |0\rangle \\ &= \frac{N}{\sqrt{2\omega'}} e^{ik'x - i\omega't} \end{aligned}$$

The next step is to repeat this derivation for a state that is defined as weighted integral of position eigenstates so we can directly relate the QM wavefunction to the QFT field.

Let $|\chi\rangle = \int_{-\infty}^{\infty} \chi(x') |x'\rangle dx'$. Then

$$\begin{aligned} \langle 0 | \hat{\phi} | \chi \rangle &= \langle 0 | \int_{-\infty}^{\infty} \frac{dk}{2\pi\sqrt{2\omega}} \hat{a}(k) e^{ikx - i\omega t} \int_{-\infty}^{\infty} dx' \chi(x') |x'\rangle \\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi\sqrt{2\omega}} e^{ikx - i\omega t} \int_{-\infty}^{\infty} \chi(x') \langle k | x' \rangle dx' \end{aligned}$$

By Sakurai (1.7.32), $\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$

$$\begin{aligned} &= \int_{-\infty}^{\infty} \frac{dk}{2\pi\sqrt{2\omega}} e^{ikx - i\omega t} \int_{-\infty}^{\infty} \chi(x') \frac{1}{\sqrt{2\pi}} e^{-ikx'} dx' \\ &= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \chi(x') \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\sqrt{2\omega}} e^{ik(x-x')} dk dx' \end{aligned}$$

The problem here is that ω depends on k , so we cannot integrate algebraically. But if we pretend that ω is independent of k , then the result is $\psi(x, t) = \frac{1}{\sqrt{2\pi}} \frac{e^{-i\omega t}}{\sqrt{2\omega}} \chi(x)$.

4 Field Quantization

Motivation is to accommodate variable numbers of particles (See quote 17).

So far everything has been classical except for the motivation for the Lagrangian; the coordinates are not yet operators. But now is the time to quantize the system by analogy with normal quantum mechanics. We now use $\hat{\pi}(\mathbf{x}) = -i\hbar \frac{\delta}{\delta\hat{\phi}(\mathbf{x})}$ so that

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2\rho} \frac{\delta^2}{\delta\hat{\phi}(\mathbf{x})^2} + \frac{1}{2}\rho c^2 (\nabla\hat{\phi})^2 + \frac{1}{2}\rho \frac{m^2 c^4}{\hbar^2} \hat{\phi}^2(\mathbf{x})$$

Question: Can we derive the expression for the momentum operator from the commutation relations?

5 Miscellaneous

Question of whether the spring constants in different directions are different. Notice that the spring constant depends on the mass of the point masses. If localized particles correspond to oscillations in this lattice, then why does a particle in motion keep going in a straight line? Well if you have a whole wavefront, then there is no lateral torsion except for at the edges of the front. If the front is wide enough, then maybe it could have enough energy to not dissipate for a while.

So how do you get particles traveling at different velocities? Is it governed by the magnitude of the oscillation? A larger oscillation would create a steeper slope, which would generate a larger force on the neighboring point masses and perhaps make the wave travel faster. Research Question: How do you theoretically calculate the phase velocities of ripples in a mattress like this? Also, does this mattress exhibit the correct uncertainty principle?

If we assume that the spring constant of the spring underneath the mass is dominant, then the frequency of oscillation will be fixed at $\omega = \sqrt{k_v/m}$. But the slope will depend on how different the amplitude is at points slightly downstream. This depends on the wavelength of the wavepacket. That gives some intuition for DeBroglie's relation. In this model, it makes perfect sense that a very long wavelength will produce slowly travelling waves because there is very little force in the tangential springs.

Maybe it just so happened that the probability density is proportional to the amplitude of oscillations. Why would it have to be complex?

Per Kraus told me that the masses in the mattress are quantum mechanical oscillators themselves, so they have a nonzero ground state energy, which is related to vacuum fluctuations. I do not yet see why it is necessary for the masses to be quantum oscillators, so I am still skeptical.

Tomboulis says there is no discretization in the primitive universe before bound states. I would like a section on all the interactions that occurred in this primitive universe: Gravitational attraction, Compton scattering, Beta Decay, Inverse Beta Decay, electron-electron scattering, electron-proton scattering, electron-positron annihilation, proton-antiproton scattering, photons collide and create particles, electron-neutrino scattering via the weak force.

6 Concepts

The point of this section is to provide a complete conceptual explanation of free quantum field theory with all of the calculations omitted. Equations will only be written down to look at, and not derived. As many statements as possible will refer to a quotation written at the end by a raised symbol: ^{Q1}.

(a) The Lagrangian density

$$\mathcal{L} = \frac{1}{2} \left[\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2 \right]$$

is defined so that its Euler-Lagrange equation of motion is the Klein-Gordon equation of quantum mechanics. (See Reference 12)

- (b) There is a similar, but slightly more generalized Lagrangian density corresponding to the Dirac equation.
- (c) The Klein-Gordon Lagrangian density represents a lattice of continuously coupled oscillators with an additional oscillator at each point in space
- (d) Teller has an awesome example: two ripples of equal amplitude traveling in opposite directions along a rope cannot be distinguished when they overlap, and you can't say whether the two ripples bounced off each other or traveled through each other unimpeded.

7 Questions

- (a) Show that the definition of Lorentz Transformations in terms of the metric is equivalent to the definition given in Griffith's electromagnetism book.
- (b) Ryder Pg 109: Why does a symmetry/gauge group being abelian determine whether the field acts as its own source? Note that in an abelian group the commutator is always zero.
- (c) In what sense is the Klein-Gordon equation applicable to reality?
- (d) Are generators in quantum mechanics related to the generators of groups in group theory?
- (e) To what extent are virtual particles real? Are they just a math trick that mirrors the terms of the Dyson series expansion?
- (f) How do you represent a static electric field with photons?
- (g) Why do we decompose $\hat{\phi}(x)$ into a superposition of momentum eigenstates? Is it just because momentum eigenstates have fixed energy and energy is conserved?
- (h) What is the justification or motivation for second quantization?
- (i) If particles are like ripples in a pond, then why don't particles disperse like ripples in a pond? Is it because the spring constant for the non-gradient springs is much larger?
- (j) What would it mean if there was a sharp spike in $\phi(x)$ at location x_0 ? Would that mean there is a particle there?
- (k) Do particles look like truncated plane waves in the field $\phi(x)$?
- (l) Can our interpretation of QFT answer the question of why the cross section of a photon is proportional to its wavelength?

8 References

1. Morse and Feshbach Part I Page 304:

Consequently the wave equation for the string corresponds to the requirement that *the difference between the total kinetic energy of the string and its potential energy be as small as possible*, on the average and subject to the initial and boundary conditions. This is a very suggestive and useful result, from which many other relations can be derived.

If, for instance, a transverse force $F(x)$ per unit length of string is applied, an additional term $-F\psi$ should be added to the potential energy; or if the string is embedded in an elastic medium (as discussed on page 139), the added term is $\frac{1}{2}K\psi^2$. The resulting Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\rho \left(\frac{\partial\psi}{\partial t} \right)^2 - \frac{1}{2}T \left(\frac{\partial\psi}{\partial x} \right)^2 - \frac{1}{2}K\psi^2 + F\psi$$

and the equation of motion is

$$\rho \frac{\partial^2 \psi}{\partial t^2} - T \frac{\partial^2 \psi}{\partial x^2} = F - K\psi$$

which corresponds to Eq. (2.1.27).

The derivative of \mathcal{L} with respect to $\dot{\psi}$ should correspond to the momentum in particle dynamics. The corresponding density

$$p = \partial \mathcal{L} / \partial \dot{\psi} = \rho (\partial \psi / \partial t)$$

will be called the *canonical momentum density* of the field at x . In the case of the string, as we see, it is the momentum of a unit length of string when moving with a velocity $[\partial \psi(x) / \partial t]$.

2. Road to Reality Page 657:

One of the earliest ideas for QFT was the procedure which goes under the rather misleading name of ‘second quantization’. According to this procedure, we try to pretend that the wavefunction ψ of some particle itself becomes an ‘operator’, acting on some shadowy state vector which may be denoted by $|0\rangle$, hiding over on the far right. ... It does not now represent the actual quantum state, but it describes the operation that ‘creates’ a new particle having this given wavefunction ψ , introducing it into the state that is there previously. ... Such an operator is referred to as a *creation operator*.

3. Goldstein 3rd Edition Page 585:

In the momentum or plane wave representations, the fields ϕ and ϕ^* are thus replaced by discrete systems of harmonic oscillators, much in the same manner that the sound field in a solid is looked on as a collection of “phonons.” The discrete spectrum of “vibrations” of our scalar charged field is given by Eq. (13.106). Quantization of the field (that is, the so-called second quantization) is done most simply via the momentum representation. In effect, the motion of each harmonic oscillator is quantized as would be done for an actual harmonic oscillator. But this subject certainly lies outside our province.

4. Goldstein 3rd Edition Page 576 (after discussing Fourier transforms):

Finally, we have introduced a description of the continuous system in terms of a denumerable, discrete set of coordinates that obey the same type of mechanics as the discrete system we started with. Because of the formal correspondence with the variables of discrete systems, the q_k and p_k quantities are the obvious candidates for quantization when we go from classical to quantum field theory. Indeed, the q_k correspond to what are spoken of as the “occupation numbers” for the field.

5. A Pedestrian Approach to Quantum Mechanics Page 47:

In the preceding chapters we have seen how the classical radiation field assumed characteristics describable in particle language when the electromagnetic field was quantized. This suggests the possibility that all of the particles found in nature may be considered as the quanta of some field. But what field? A natural choice is the wave function $\psi(\mathbf{x}, t)$ which describes the particle.

6. Carroll Page 40:

Upon quantization, excitations of the field are observable as particles. Scalar fields give rise to spinless particles, while vector fields and other tensors give rise to higher-spin particles. If the field were complex instead of real, it would have two degrees of freedom rather than just one, which would be interpreted as a particle and a distinct antiparticle. Real fields are their own antiparticles. An example of a real scalar field would be the neutral π -meson.

7. Carroll Page 41:

You may be wondering how a field can have mass. When we quantize the field we find that momentum eigenstates are collections of particles, each with mass m . At the classical level, we think of “mass” as simply a convenient characterization of the field dynamics.

8. Carroll Page 43:

You may wonder what the purpose of introducing a Lagrangian formulation is, if we were able to invent the equations of motion before we ever knew the Lagrangian (as Maxwell did for his equations). There are a number of reasons, starting with the basic simplicity of positing a single scalar function of spacetime, the Lagrangian density, rather than a number of (perhaps tensor-valued) equations of motion. Another reason is the ease with which symmetries are implemented; demanding that the action be invariant under a symmetry ensures that the dynamics respect the symmetry as well. Finally, as we will see in Chapter 4, the action leads via a direct procedure (involving varying with respect to the metric itself) to a unique energy-momentum tensor.

9. Carroll Page 44:

Why should we consider one classical field theory rather than some other one ... how should we pick a Lagrangian for this field? ... Ultimately, of course, we work by trial and error and try to fit the data given to us by experiment. ... Only the lowest order terms [in effective field theory] we can put in our Lagrangian will come with dimensionless couplings (or ones with units of mass to a positive power), so we only need to bother with those at low energies. This feature of field theory allows for a dramatic simplification in considering all of the models we might want to examine.

10. Elementary Quantum Field Theory Page 24:

In the mechanical model of a displacement field which we had in mind so far, the general three-dimensional case is somewhat more complicated, since the field is then a vector and has three components. However, by only allowing displacements of a three-dimensional atomic lattice in one direction, say x , as shown in Fig 4.1, we have the discrete analogue of a scalar (hermitian) field $\phi(x, y, z)$. It is this simpler case which will prove to be an appropriate description for pions...

11. Elementary Quantum Field Theory Page 31:

The use of the mechanical analogy to find the field equations (4.6) may not seem very convincing when applied to, say, the pion field. To do this more systematically, the Lagrangian formalism is essential. To satisfy Lorentz invariance, the Lagrangian density, e.g., the quantity under the integral in (4.20), has to be a scalar. Our expression is, in fact, the most general scalar which is quadratic in the field and its derivatives.

$$L(t) = \int d^3r \frac{1}{2} [\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2] \quad (4.20)$$

12. Morse and Feshbach Part I Page 344:

Klein-Gordon Equation

Lagrange Density $\mathcal{L} = -\frac{\hbar^2}{2m} \left[(\nabla\psi^*)(\nabla\psi) - \frac{1}{c^2} \left(\frac{\partial\psi^*}{\partial t} \right) \left(\frac{\partial\psi}{\partial t} \right) + \left(\frac{mc^2}{\hbar} \right) \psi^*\psi \right]$

Lagrange-Euler equation for ψ , $\nabla^2\psi - \frac{1}{c^2} \left(\frac{\partial^2\psi}{\partial t^2} \right) = \left(\frac{mc}{\hbar} \right)^2 \psi$ (Klein-Gordon equation).

12. Morse and Feshbach Part I Page 316:

Here again we use two independent field variables, ψ and ψ^* .

13. Goldstein 3rd Edition Page 583:

Accordingly, the Lagrangian density and associated functions will here be given in terms of two independent field variables, ϕ and ϕ^* , each of which are 4-scalar.

14. Peskin and Schroeder Page 79:

In fact, we will not discuss canonical quantization of the electromagnetic field at all in this book. It is an awkward subject, essentially because of gauge invariance.

15. Teller Page 68:

Early in the study of relativistic wave equations it was felt that the problems with the Klein-Gordon equation stemmed, at least in part, from the fact that it was second order in time. Accordingly, Dirac set out to formulate a wave equation that was both first order in time and free of the square root of a differential operator. This could be done if one could somehow express

$$E = \omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$$

as a linear function of \mathbf{p} , for then, after the heuristic substitution of (4.9) and (4.10) one would have a wave equation linear in both $\frac{\partial}{\partial t}$ and ∇_x . In other words, we would like to be able to find an $\alpha_1, \alpha_2, \alpha_3$, and β such that

$$\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta m = \sqrt{\mathbf{p}^2 + m^2}.$$

Dirac noticed that one could have $(\sum_i \alpha_i p_i + \beta m)^2 = \mathbf{p}^2 + m^2$ by taking the α_i and β to be 4×4 matrices satisfying the relations

$$\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = \hat{I}$$

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 0, \quad i, j = 1, 2, 3; i \neq j$$

and

$$\beta \alpha_i + \alpha_i \beta = 0, \quad i = 1, 2, 3.$$

Following the by-now familiar procedure of using the heuristic substitutions, this time into (4.11) and (4.25), to formulate a wave equation, we get the Dirac equation

$$\left(i \frac{\partial}{\partial t} + i \boldsymbol{\alpha} \cdot \nabla_x - \beta m \right) \Psi(\mathbf{x}, t) = 0,$$

with $\boldsymbol{\alpha}$ the three-component “vector” with matrix components $\alpha_1, \alpha_2, \alpha_3$. Because the α_i and β are 4×4 matrices, $\Psi(\mathbf{x}, t)$ must be taken to be a four-component “vector,” called a (Dirac, or four-component) spinor, with components $\Psi_i(\mathbf{x}, t)$, $i = 1, 2, 3, 4$.

16. Teller Page 167: (Verbose!)

In particular, by measuring the mass and charge we make it unnecessary to be able accurately to evaluate those integrals that in our current approximation we can't even get to come out convergent.

17. Wikipedia - Canonical Quantization:

In quantum mechanics one deals with the quantum states of a system of a fixed number of particles. This is inadequate for the study of systems in which particles are created and destroyed. Historically, this problem was solved through the introduction of quantum field theory.