QFT Field

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1 Second Quantization

The motivation for second quantization comes from the need for variable numbers of particles. Let $\hat{\phi}(\mathbf{x}, t)$ be the operator whose eigenvalues correspond to solutions to the Klein-Gordon equation. We can write it in terms of its Fourier transform

$$\hat{\phi}(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \hat{\phi}(\mathbf{p}, t)$$

So we would like to determine $\hat{\phi}(\mathbf{p}, t)$, which should satisfy the Klein-Gordon equation in momentum space

$$\left[\frac{\partial^2}{\partial t^2} + (|\mathbf{p}^2| + m^2)\right] \hat{\phi}(\mathbf{p}, t) = 0$$

$$\left[\frac{\partial^2}{\partial t^2} + \omega_{\mathbf{p}}^2\right] \hat{\phi}(\mathbf{p}, t) = 0$$

where $\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}^2| + m^2}$. We recognize the similarity between this and the equation of motion of a simple harmonic oscillator

$$\ddot{x} + \omega^2 x = 0$$

which can be converted into an operator equation

$$\left[\frac{\partial^2}{\partial t^2} + \omega^2\right] \hat{x} = 0$$

This has the same meaning as the previous equation, except that it must be applied to a state in order to form a real equation. We know from quantum mechanics that we can write $\hat{x}$ as

$$\hat{x} = \frac{1}{\sqrt{2\omega}} \left(\hat{a} + \hat{a}^\dagger\right)$$

So we have

$$\left[\frac{\partial^2}{\partial t^2} + \omega^2\right] \left[\frac{1}{\sqrt{2\omega}} \left(\hat{a} + \hat{a}^\dagger\right)\right] = 0$$

Now we can turn this into an infinite set of equations, one for each $\mathbf{p}$ by making $\omega, a, a^\dagger$ functions of $\mathbf{p}$.

$$\left[\frac{\partial^2}{\partial t^2} + \omega(\mathbf{p})^2\right] \left[\frac{1}{\sqrt{2\omega(\mathbf{p})}} \left(\hat{a}(\mathbf{p}) + \hat{a}^\dagger(\mathbf{p})\right)\right] = 0$$

By comparing this with the Klein-Gordon operator equation we may guess that

$$\hat{\phi}(\mathbf{p}, t) \stackrel{?}{=} \frac{1}{\sqrt{2\omega(\mathbf{p})}} \left(\hat{a}(\mathbf{p}) + \hat{a}^\dagger(\mathbf{p})\right)$$

But we need to be careful if we want $\phi(\mathbf{x}, t)$ to be real. In that case we require $\phi(\mathbf{x}, t) = \phi^*(\mathbf{x}, t)$, which is satisfied by $\phi^*(\mathbf{p}, t) = \phi(-\mathbf{p}, t)$. So we want to negate the momentum in one of the a parameters. We choose the second

$$\hat{\phi}(\mathbf{p}, t) = \frac{1}{\sqrt{2\omega(\mathbf{p})}} \left(\hat{a}(\mathbf{p}) + \hat{a}^\dagger(-\mathbf{p})\right)$$

Therefore we conclude that [1]

$$\hat{\phi}(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\mathbf{p})}} e^{i\mathbf{p}\cdot\mathbf{x}} \left(\hat{a}(\mathbf{p}) + \hat{a}^\dagger(-\mathbf{p})\right)$$

[1] See Peskin and Schroeder (2.27)

2 Field-Wavefunction Relation Using Position Eigenstates

A state containing a single particle with wave-function $\psi(x)$ is given by [2]

$$|\Psi_1\rangle = \int d^3x' \ \psi(x')\hat{\phi}^*(x')\,|0\rangle$$

Therefore we compute the position representation of the QFT field in analogy with quantum mechanics by applying a position eigenstate.

$$\Psi_1(x) = \langle x|\ \Psi_1\rangle = \int d^3x' \ \left\langle x\middle|\,\psi(x')\hat{\phi}^*(x')\,\middle|0\right\rangle$$

$$= \int d^3x' \ \left\langle x\middle|\,\psi(x')\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\mathbf{p})}}e^{-i\mathbf{p}\cdot\mathbf{x}'}\hat{a}^\dagger(\mathbf{p})\,\middle|0\right\rangle$$

$$= \int d^3x' \ \left\langle x\middle|\,\psi(x')\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\mathbf{p})}}\frac{1}{(2\pi)^{3/2}}\,\langle p|\ x'\rangle\,\hat{a}^\dagger(\mathbf{p})\,\middle|0\right\rangle$$

$$= \int d^3x' \ \langle x|\ x'\rangle\,\psi(x')\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\mathbf{p})}}\frac{1}{(2\pi)^{3/2}}\left\langle p\middle|\,\hat{a}^\dagger(\mathbf{p})\,\middle|0\right\rangle$$

$$= \psi(x)\int \frac{d^3p}{\sqrt{2\omega(\mathbf{p})}}\frac{1}{(2\pi)^{3/2}}\left\langle p\middle|\,\hat{a}^\dagger(\mathbf{p})\,\middle|0\right\rangle$$

$$= \psi(x)\int \frac{d^3p}{2\omega_{\mathbf{p}}}\frac{1}{(2\pi)^{3/2}}\,\langle p|\ \psi_p\rangle$$

using the equation $|\psi_p\rangle = \sqrt{2\omega(\mathbf{p})}a^\dagger(\mathbf{p})\,|0\rangle$.[3] But this is bad because it means

$$\Psi_1(x) = \infty\cdot\psi(x)$$

3 Field-Wavefunction Relation Using Localized Particle States

In analogy with quantum mechanics, we can find the field itself by applying a single particle state localized at x, which we denote by $|\psi_x\rangle$

$$\Psi_1(x) = \langle\psi_x|\ \Psi_1\rangle = \int d^3x' \ \left\langle\psi_x\middle|\,\psi(x')\hat{\phi}^*(x')\,\middle|0\right\rangle$$

$$= \int d^3x' \ \langle\psi_x|\,\psi(x')\,|\psi_{x'}\rangle = \int d^3x' \ \psi(x')\,\langle\psi_x|\ \psi_{x'}\rangle$$

$$= \int d^3x' \ \psi(x')\delta(x-x') = \psi(x)$$

Therefore, for a single particle, the QFT field is the QM field. Next we consider the case of two particles.

$$\Psi_2(x) \stackrel{?}{=} \langle\psi_x|\ \Psi_2\rangle = \int\int d^3x' \ d^3x'' \ \psi_1(x')\psi_2(x'')\,\langle\psi_x|\ \psi_{x'},\psi_{x''}\rangle$$

But the bracket will always be zero because of the following argument. For example, consider the case

$$\langle\psi_x|\ \psi_x,\psi_y\rangle = \left\langle 0\middle|\,\hat{\phi}(x)\hat{\phi}^*(x)\hat{\phi}^*(y)\,\middle|0\right\rangle = \langle 0|\ \psi_y\rangle = 0$$

[2] See Hatfield (2.69)

[3] See Peskin and Schroeder (2.35)

In general, if there are any odd number of creation and annihilation operators in a ground state expectation value, then the result will be zero by orthogonality of states. So the projection with the set of single particle position eigenstates always gives zero. The analogy with quantum mechanics does not apply directly to give the field in the position representation. The problem is that we need to project with a state that does not care how many other particles there are in the universe. We define

$$\langle \psi_x^* | \equiv \langle \psi_x | + \int d^3z \ \langle \psi_x, \psi_z | + \int \int d^3z \ d^3z' \ \langle \psi_x, \psi_z, \psi_{z'} | + \cdots$$

Then we can define $\Psi_2(x)$ by

$$\Psi_2(x) \equiv \langle \psi_x^* | \ \Psi_2 \rangle = \int \int \int d^3z \ d^3x' \ d^3x'' \ \psi_1(x')\psi_2(x'') \langle \psi_x, \psi_z | \ \psi_{x'}, \psi_{x''} \rangle$$

$$= \int \int \int d^3z \ d^3x' \ d^3x'' \ \psi_1(x')\psi_2(x'')[\delta(x-x')\delta(z-x'') + \delta(x-x'')\delta(z-x')]$$

$$= \int \int d^3x' \ d^3x'' \ \psi_1(x')\psi_2(x'')[\delta(x-x') + \delta(x-x'')]$$

$$= \int \int d^3x' \ d^3x'' \ \psi_1(x')\psi_2(x'')[\delta(x-x') + \delta(x-x'')]$$

$$= \psi_1(x) \int d^3x'' \ \psi_2(x'') + \psi_2(x) \int d^3x' \ \psi_1(x')$$

Therefore, under this definition, the QFT field containing two particles is a linear combination of the QM wave-functions of the particles,

$$\Psi_2(x) = c_1(t)\psi_1(x) + c_2(t)\psi_2(x)$$

4 Questions

- How do you derive Hatfield (2.69) and why does it contain the wave-function instead of the probability (magnitude squared of wave-function)?
- Why does $\hat{\phi}$ in the first section have to be real?