10. *Electricity and Magnetism* (Fall 2002)

A point charge q is **inside** a hollow, grounded, conducting sphere of inner radius a. Use the method of images to find

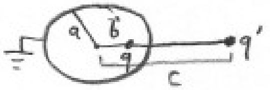
(a) the potential inside the sphere;

(b) the induced surface-charge density at the point on the sphere nearest to q [Editor's Note: You may assume that the outer radius is different from the inner radius so the sphere is not an infinitesimal shell.];

(c) the magnitude and direction of the force acting on q.

(d) Is there any change in the solution if the sphere is kept at a fixed potential V? If the sphere has a **total** charge Q on its inner and outer surface?

a.



We have the two unknowns q' and c which we determine by applying the two constraints of zero potential at the nearest and farthest points.

For a point charge $V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$

$$V(a) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{a-b} + \frac{q'}{c-a}\right) = 0 \Rightarrow \frac{q'}{q} = -\frac{c-a}{a-b}$$

$$V(-a) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{a+b} + \frac{q'}{c+a}\right) = 0 \Rightarrow \frac{q'}{q} = -\frac{c+a}{a+b}$$

$$\Rightarrow \frac{c-a}{a-b} = \frac{c+a}{a+b} \Rightarrow (c-a)(a+b) = (c+a)(a-b)$$

$$\Rightarrow ca - a^2 + bc - \cancel{ab} = \cancel{ca} + a^2 - bc - \cancel{ab}$$

$$\Rightarrow a^2 = bc \Rightarrow \boxed{c = \frac{a^2}{b}}$$

So $\frac{q'}{q} = \frac{a-c}{a-b} = \frac{a - a^2/b}{a-b} = \frac{ab-a^2}{ab-b^2} = \frac{a(b-a)}{b(a-b)} = -\frac{a}{b}$

$$\Rightarrow \boxed{q' = -\frac{a}{b}q}$$

If $\vec{b}$ is the location of the point charge with the origin at the center, then

$$V(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\vec{x}-\vec{b}|} - \frac{\frac{a}{b}q}{|\vec{x} - \frac{a^2}{b^2}\vec{b}|}\right)$$

b. Since the electric field is zero inside the wall, the discontinuity is equal to the field just inside.

$$\Delta E = E = \frac{\sigma}{\epsilon_0} \Rightarrow \sigma(a) = -\epsilon_0 \frac{\partial V}{\partial(-x)}\Big|_{x=a} = \epsilon_0 \frac{\partial V}{\partial x}\Big|_{x=a}$$

$$V(x) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(x-b)} - \frac{a}{b}\frac{1}{(x-a^2/b)}\right) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(x-b)} - \frac{a}{(bx-a^2)}\right)$$

$$\frac{\partial V}{\partial x} = \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{(x-b)^2} + \frac{ab}{(bx-a^2)^2}\right)$$

$$\sigma(a) = \epsilon_0 \frac{\partial V}{\partial x}\Big|_{x=a} = \frac{q}{4\pi}\left(-\frac{1}{(a-b)^2} + \frac{ab}{(ab-a^2)^2}\right) = \frac{q}{4\pi}\left(\frac{b/a - 1}{(b-a)^2}\right) = \frac{q}{4\pi}\frac{1}{ab-a^2}$$

c. The force acting on q is equal to the force due to the image charge

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(c-b)^2} = \frac{1}{4\pi\epsilon_0}\frac{-q^2a/b}{(a^2/b-b)^2} = -\frac{1}{4\pi\epsilon_0}\frac{q^2ab}{(a^2-b^2)^2}$$

and the force is radially outward from the center of the sphere.

d. A fixed potential or net charge changes parts a and b, but not c.