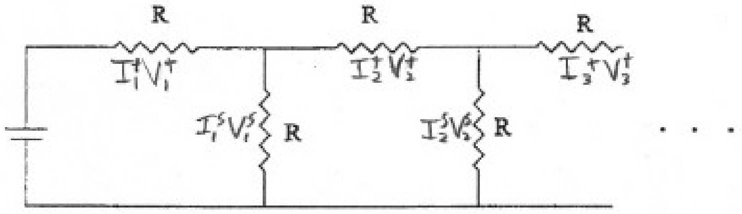
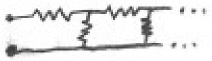
12. *Electricity and Magnetism* (Fall 2002)

A voltage is applied to the infinitely long resistor network shown below. Each resistor has the same resistance R. Calculate the power dissipated in each resistor.



First we find the total resistance of the network, $\tilde{R}$.

Let $\tilde{R}$ be the resistance of 

Then $$\tilde{R} = R + \frac{1}{\frac{1}{R}+\frac{1}{\tilde{R}}}$$

$$\Rightarrow \tilde{R}\left(\frac{1}{R}+\frac{1}{\tilde{R}}\right) = R\left(\frac{1}{R}+\frac{1}{\tilde{R}}\right)+1$$

$$\Rightarrow \frac{\tilde{R}}{R}+1 = 1+\frac{R}{\tilde{R}}+1$$

$$\Rightarrow \frac{\tilde{R}}{R} = \frac{R}{\tilde{R}}+1$$

$$\Rightarrow \tilde{R}^2 = R^2+R\tilde{R}$$

$$\Rightarrow \tilde{R}^2 - R\tilde{R} - R^2 = 0$$

$$\Rightarrow \tilde{R} = \tfrac{1}{2}\left[R \pm \sqrt{R^2+4R^2}\right]$$

$$\Rightarrow \tilde{R} = \tfrac{1}{2}R\left(1+\sqrt{5}\right) \quad \text{Since it must be pos.}$$

Let V_n^+, V_n^s be the voltage drop across the top, side resistors.

By Ohm's Law, $$I_n^+ = \frac{V_n^+}{R} = \frac{V_{n-1}^s}{\tilde{R}} \qquad (1)$$

By Kirchoff's Loop rule, $$V_n^+ + V_n^s = V_{n-1}^s \qquad (2)$$

(2)→(1): $$\frac{V_{n-1}^s - V_n^s}{R} = \frac{V_{n-1}^s}{\tilde{R}}$$

$$\Rightarrow \frac{V_n^s}{R} = \frac{V_{n-1}^s}{R} - \frac{V_{n-1}^s}{\tilde{R}} = \left(\frac{1}{R}-\frac{1}{\tilde{R}}\right)V_{n-1}^s$$

$$\Rightarrow V_n^s = \left(1-\frac{R}{\tilde{R}}\right)V_{n-1}^s$$

$$\Rightarrow V_n^s = \left(1-\frac{R}{\tilde{R}}\right)^n V$$

And by (1), $$V_n^+ = \frac{R}{\tilde{R}}V_{n-1}^s = \frac{R}{\tilde{R}}\left(1-\frac{R}{\tilde{R}}\right)^{n-1}V$$

Therefore $$P_n^s = \frac{(V_n^s)^2}{R} = \frac{V^2}{R}\left(1-\frac{R}{\tilde{R}}\right)^{2n}$$

and $$P_n^+ = \frac{(V_n^+)^2}{R} = \frac{V^2R}{\tilde{R}}\left(1-\frac{R}{\tilde{R}}\right)^{2n-2}$$