1. *Quantum Mechanics* (Fall 2003)

Two identical spin-$\frac{1}{2}$ particles interact via the Hamiltonian

$$H = J(S_1^x S_2^x + S_1^y S_2^y + k S_1^z S_2^z) + \mu(S_1^z + S_2^z)B.$$

(a) Find the energy levels of this system assuming that the particles are in an anti-symmetric spatial wavefunction.

(b) Repeat for a symmetric spatial wavefunction.

The total wavefunction is the product of the spatial and spin wavefunctions. Since electrons are fermions, they must have antisymmetric total wavefunctions, which means the spin part and the spatial part have opposite parity.

$$H = J(S_1^x S_2^x + S_1^y S_2^y + K S_1^z S_2^z) + \mu(S_1^z + S_2^z)B$$

$$= J[\vec{S}_1 \cdot \vec{S}_2 + (K-1) S_1^z S_2^z] + \mu(S_1^z + S_2^z)B$$

$$= J[\tfrac{1}{2}(\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2) + (K-1) S_1^z S_2^z] + \mu(S_1^z + S_2^z)B$$

$$= J[\tfrac{1}{2}\vec{S}^2 - \tfrac{3}{4}\hbar^2 + (K-1) S_1^z S_2^z] + \mu(S_1^z + S_2^z)B$$

since $\vec{S}_i^2$ has eigenvalue $s_i(s_i+1)\hbar^2$ and $s_i = \frac{1}{2}$

$$= J[\tfrac{1}{2}\vec{S}^2 - \tfrac{3}{4}\hbar^2 + (K-1)\tfrac{1}{2}(S_z^2 - S_{1z}^2 - S_{2z}^2)] + \mu S_z B$$

$$= J[\tfrac{1}{2}\vec{S}^2 - \tfrac{3}{4}\hbar^2 + (K-1)\tfrac{1}{2}(S_z^2 - \tfrac{\hbar^2}{2})] + \mu S_z B$$

because in an energy eigenstate $\vec{S}_1$ and $\vec{S}_2$ must be in the z-direction.

Now we see that the states $|s_1 s_2 s m\rangle$ are energy eigenstates:

$$|0\,0\rangle = \tfrac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \qquad |1\,1\rangle = |++\rangle$$

$$|1\,0\rangle = \tfrac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) \qquad |1\,-1\rangle = |--\rangle$$

and the first is the only antisymmetric one.

a. Antisymmetric spatial wavefunction $\Rightarrow$ symmetric spin wavefunction

$$E_{|11\rangle} = J[\tfrac{1}{2}(2\hbar^2) - \tfrac{3}{4}\hbar^2 + (K-1)\tfrac{\hbar^2}{4}] + \mu\hbar B = \tfrac{1}{4}\hbar^2 JK + \mu\hbar B$$

$$E_{|10\rangle} = J[\tfrac{1}{2}(2\hbar^2) - \tfrac{3}{4}\hbar^2 + (K-1)(-\tfrac{\hbar^2}{4})] + 0 = \tfrac{1}{4}\hbar^2 J(2-K)$$

$$E_{|1-1\rangle} = J[\tfrac{1}{2}(2\hbar^2) - \tfrac{3}{4}\hbar^2 + (K-1)(\tfrac{\hbar^2}{4})] - \mu\hbar B = \tfrac{1}{4}\hbar^2 JK - \mu\hbar B$$

b. Symmetric spatial wavefunction $\Rightarrow$ antisymmetric spin wavefunction

$$E_{|00\rangle} = J[\tfrac{1}{2}(0) - \tfrac{3}{4}\hbar^2 + (K-1)(-\tfrac{\hbar^2}{4})] + 0 = -\tfrac{1}{4}\hbar^2 J(K+2)$$