3. *Quantum Mechanics* (Fall 2003)

Consider a particle moving in the potential

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2 x^2 & \text{if } x > 0 \\ \infty & \text{otherwise} \end{cases}$$

(a) What is the lowest energy eigenvalue?

(b) What is $\langle x^2 \rangle$?

See Griffiths Problem 2.42

a. We assume that the solutions to the half harmonic oscillator are a subset of the solutions to the full harmonic oscillator. Only the odd solutions are permissible.

To find the lowest odd solution we use the fact that $a|\psi_0\rangle = 0$

Recall $a = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega x + ip)$

$$0 = a|\psi_0\rangle = \frac{1}{\sqrt{2m\omega\hbar}}\left(m\omega x|\psi_0\rangle + \hbar\frac{\partial}{\partial x}|\psi_0\rangle\right)$$

$$\Rightarrow \frac{\partial}{\partial x}|\psi_0\rangle = -\frac{m\omega}{\hbar}x|\psi_0\rangle$$

$$\Rightarrow |\psi_0\rangle = Ae^{-\frac{m\omega}{2\hbar}x^2} \quad \text{which is even}$$

$$|\psi_1\rangle = a^\dagger|\psi_0\rangle = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega x - ip)|\psi_0\rangle$$

$$\propto m\omega x e^{-\frac{m\omega}{2\hbar}x^2} - \hbar\frac{\partial}{\partial x}e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= m\omega x e^{-\frac{m\omega}{2\hbar}x^2} + m\omega x e^{-\frac{m\omega}{2\hbar}x^2} \quad \text{which is odd}$$

So the lowest energy eigenvalue is $E_1 = (1+\frac{1}{2})\hbar\omega = \frac{3}{2}\hbar\omega$

b. $|\psi_1\rangle$ is odd so $|\langle\psi_1|\psi_1\rangle|^2$ is even which means the probability distribution for x is the same on both sides of zero for the full SHO. Therefore the standard deviation of x won't be affected by considering only the positive side. Therefore we can just calculate $\langle x^2\rangle$ for the full SHO.

$$\langle\psi_1|x^2|\psi_1\rangle = \langle\psi_1|\frac{\hbar}{2m\omega}(a^\dagger+a)^2|\psi_1\rangle$$

$$= \frac{\hbar}{2m\omega}\langle\psi_1|\cancel{(a^\dagger)^2} + a^\dagger a + aa^\dagger + \cancel{a^2}|\psi_1\rangle$$

$$= \frac{\hbar}{2m\omega}\langle\psi_1|1+2|\psi_1\rangle$$

$$= \frac{3\hbar}{2m\omega}$$