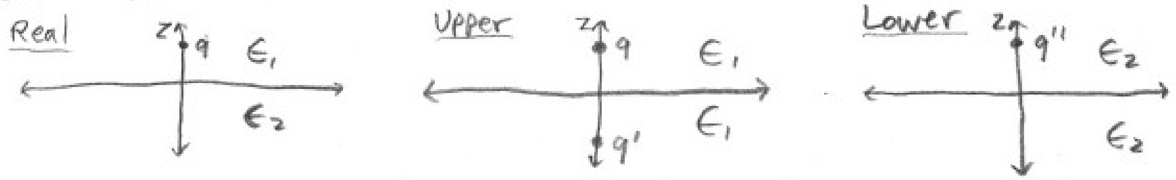
10. *Electricity and Magnetism* (Fall 2004)

The upper half-space is filled with a material of permittivity ϵ_1, while the lower half space is filled with a different material with permittivity ϵ_2. Their interface is located at the $z = 0$ plane. A point charge q is located at $\mathbf{r}_q = d\hat{\mathbf{z}}$ on the z-axis in medium 1. Find the electrostatic potential everywhere.

See Jackson Page 154

For this problem you have to know the trick that you can satisfy the Laplace/Poisson equation and the boundary conditions for the upper half space by replacing the ϵ_2 medium with an ϵ_1 medium and a point charge in the image location, and for the lower half space by replacing the ϵ_1 medium with an ϵ_2 medium and a point charge on top of q.



Note: This is not a direct consequence of the method of images.

So $\Phi = \begin{cases} \Phi_1 \equiv \frac{1}{4\pi\epsilon_1}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right) & z>0 \\ \Phi_2 \equiv \frac{1}{4\pi\epsilon_2}\left(\frac{q''}{R_1}\right) & z<0 \end{cases}$ where $R_1 = \sqrt{r^2+(z-d)^2}$, $R_2 = \sqrt{r^2+(z+d)^2}$

Subject to the boundary conditions $\vec{E}^2_{\parallel} - \vec{E}^1_{\parallel} = 0$, $\vec{D}^2_{\perp} - \vec{D}^1_{\perp} = \sigma_f = 0$

which we can express as $\lim_{z\to 0^+}\begin{Bmatrix} E_x \\ E_y \\ \epsilon_1 E_z \end{Bmatrix} = \lim_{z\to 0^-}\begin{Bmatrix} E_x \\ E_y \\ \epsilon_2 E_z \end{Bmatrix}$

To facilitate the calculation we can observe that

$$\frac{\partial}{\partial z}\left(\frac{1}{R_1}\right)\bigg|_{z=0} = -\frac{\partial}{\partial z}\left(\frac{1}{R_2}\right)\bigg|_{z=0} \equiv A$$

$$\frac{\partial}{\partial r}\left(\frac{1}{R_1}\right)\bigg|_{z=0} = \frac{\partial}{\partial r}\left(\frac{1}{R_2}\right)\bigg|_{z=0} \equiv B$$

Now $\vec{E}^1_{\parallel} = \vec{E}^2_{\parallel} \Rightarrow -\frac{\partial\Phi_1}{\partial r}\Big|_{z=0} = -\frac{\partial\Phi_2}{\partial r}\Big|_{z=0} \Rightarrow \frac{-1}{4\pi\epsilon_1}(qB + q'B) = \frac{-1}{4\pi\epsilon_2}q''B$

$$\Rightarrow \frac{q+q'}{\epsilon_1} = \frac{q''}{\epsilon_2}$$

and $\vec{D}^1_{\perp} = \vec{D}^2_{\perp} \Rightarrow -\epsilon_1\frac{\partial\Phi_1}{\partial z}\Big|_{z=0} = -\epsilon_2\frac{\partial\Phi_2}{\partial z}\Big|_{z=0}$

$$\Rightarrow \frac{-1}{4\pi}(qA - q'A) = -\frac{1}{4\pi}q''A \Rightarrow q - q' = q''$$

Combining, $\frac{q+q'}{\epsilon_1} = \frac{q-q'}{\epsilon_2} \Rightarrow q'\left(\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}\right) = q\left(\frac{1}{\epsilon_2}-\frac{1}{\epsilon_1}\right)$

$$\Rightarrow q'(\epsilon_2+\epsilon_1) = q(\epsilon_1-\epsilon_2) \Rightarrow q' = \frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}q$$

and $q'' = q - q' = \left(1 - \frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\right)q = \frac{2\epsilon_2}{\epsilon_1+\epsilon_2}q$

Therefore $\Phi = \begin{cases} \frac{1}{4\pi\epsilon_1}\left(\frac{q}{\sqrt{r^2+(z-d)^2}} + \left(\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\right)\frac{q'}{\sqrt{r^2+(z+d)^2}}\right) & z>0 \\ \frac{1}{4\pi(\epsilon_1+\epsilon_2)}\frac{2q}{\sqrt{r^2+(z-d)^2}} & z<0 \end{cases}$