12. *Electricity and Magnetism* (Fall 2004)

(a) Show that the annihilation of an electron and a positron can produce a single massive particle (say, X), but not a single photon.

(b) A positron beam of energy E can be made to annihilate against electrons by hitting electrons at rest in a fixed-target machine or by hitting electrons moving in the opposite direction with the same energy E in an electron-positron collider (colliding-beam accelerator). Show that the minimum energy E_{min} of a positron beam needed to produce neutral particles X of mass $M \gg m_e$ (where m_e is the electron rest mass) is much greater in a fixed-target machine than in a collider.

a. If an electron and a positron were to annihilate into a single photon, it would be impossible to conserve momentum in all frames: in the center of mass frame where the total momentum is zero, but the photon created must have nonzero momentum. A massive particle can still be created because it can be stationary in the center of mass frame.

b. We want to find the total energy in the center of mass frame $E_{tot}^{cm} = 2E_1^{cm}$ so that we don't have to take into account leftover kinetic energy required by momentum conservation. Then E_{min} is the value of E_1^{lab} in the lab frame when $E_{tot}^{cm} = Mc^2 \Leftrightarrow E_1^{cm} = \frac{1}{2}Mc^2$

For a collider, we are already in the CM frame, so $E_1^{lab} = E_1^{cm}$

$$E_{min}^{col} = E_1^{lab}\Big|_{E_1^{cm}=\frac{1}{2}Mc^2} = E_1^{cm}\Big|_{E_1^{cm}=\frac{1}{2}Mc^2} = \frac{1}{2}Mc^2$$

For a fixed target, $E_1^{lab} \neq E_1^{cm}$. If you start from the center of mass where each particle has velocity u_{cm}, say, then the velocity of the projectile in the lab frame, u_{lab}, is

$$u_{lab} = \frac{2u_{cm}}{1 + u_{cm}^2/c^2} \quad \left(\text{from} \quad u' = \frac{u+v}{1+uv/c^2}\right)$$

since you are adding the velocities of the particles relativistically.

$$\text{So } E_1^{lab} = \gamma_{lab} mc^2 = \frac{mc^2}{\sqrt{1-\beta_{lab}^2}} = \frac{mc^2}{\sqrt{1 - \frac{4\beta_{cm}^2}{(1+\beta_{cm}^2)^2}}} = \frac{mc^2}{\sqrt{\frac{1-2\beta_{cm}^2+\beta_{cm}^4}{(1+\beta_{cm}^2)^2}}}$$

$$= mc^2\sqrt{\frac{(1+\beta_{cm}^2)^2}{(1-\beta_{cm}^2)^2}} = mc^2\gamma_{cm}^2(1+\beta_{cm}^2) = mc^2\gamma_{cm}^2\left(1+1-\frac{1}{\gamma_{cm}^2}\right)$$

$$= 2mc^2\gamma_{cm}^2 - mc^2 = \frac{2(E_1^{cm})^2}{mc^2} - mc^2$$

$$\text{Therefore } E_{min}^{fix} = E_1^{lab}\Big|_{E_1^{cm}=\frac{1}{2}Mc^2} = \frac{2}{mc^2}\left(\frac{1}{2}Mc^2\right)^2 - mc^2 = \frac{1}{2}\frac{M^2}{m}c^2 - mc^2$$

$$\Rightarrow E_{min}^{fix} \cong \frac{1}{2}\frac{M^2}{m}c^2 \gg \frac{1}{2}Mc^2 = E_{min}^{col}$$

since $M \gg m$