4. *Quantum Mechanics* (Fall 2004)

Let H be the Hamiltonian for the hydrogen atom, including spin. $\hbar\mathbf{L} = \mathbf{r}\times\mathbf{p}$ and $\hbar\mathbf{s}$ are the orbital and spin angular momentum, respectively, and $\mathbf{J} = \mathbf{L} + \mathbf{s}$. Conventionally, the states are labeled $|n,l,j,m\rangle$ and they are eigenstates of H, $\mathbf{L}^2$, $\mathbf{J}^2$, and J_z.

(a) If the electron is in the state $|n,l,j,m\rangle$, what values will be measured for these four observables in terms of $\hbar$, c, the fine-structure constant α, and the electron mass m?

(b) What are the restrictions on the possible values of n, l, j, and m?

(c) Let $\mathbf{J}_\pm = J_x \pm iJ_y$. What are Recall $J_\pm |n\,\ell\, j\, m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,|n\,\ell\, j\, m\pm1\rangle$

(i) $\langle 3,1,\frac{3}{2},\frac{3}{2}|J_+|3,1,\frac{3}{2},-\frac{1}{2}\rangle = ?$ $\sqrt{\frac{15}{4}+\frac{1}{4}}\,\langle 3,1,\frac{3}{2},\frac{3}{2}|3,1,\frac{3}{2},\frac{1}{2}\rangle = 0$ by orthogonality

(ii) $\langle 3,1,\frac{3}{2},\frac{3}{2}|J_+|3,1,\frac{3}{2},\frac{1}{2}\rangle = ?$ $\sqrt{\frac{15}{4}-\frac{3}{4}}\,\langle 3,1,\frac{3}{2},\frac{3}{2}|3,1,\frac{3}{2},\frac{3}{2}\rangle = \sqrt{\frac{12}{4}} = \sqrt{3}$

(iii) $\langle 2,1,\frac{3}{2},\frac{3}{2}|p_z|2,1,\frac{3}{2},\frac{1}{2}\rangle = ?$ See below

(iv) $\langle 2,1,\frac{1}{2},-\frac{1}{2}|\mathbf{L}^2|2,1,\frac{1}{2},-\frac{1}{2}\rangle = ?$ $1(1+1) = 2$

(v) $\langle 3,2,\frac{3}{2},-\frac{1}{2}|\mathbf{J}^2|3,2,\frac{3}{2},-\frac{1}{2}\rangle = ?$ $\frac{3}{2}(\frac{3}{2}+1) = \frac{15}{4}$

(vi) $\langle 3,1,\frac{3}{2},\frac{3}{2}|J_z|3,1,\frac{3}{2},\frac{1}{2}\rangle = ?$ $\frac{1}{2}\langle 3,1,\frac{3}{2},\frac{3}{2}|3,1,\frac{3}{2},\frac{1}{2}\rangle = 0$ by orthogonality

(d) What is $\langle 1,0,\frac{1}{2},\frac{1}{2}|p_i p_j|1,0,\frac{1}{2},\frac{1}{2}\rangle = ?$

(e) For given n, l, j, and m, what are the conditions on n', l', j', and m' so that

$$\langle n',l',j',m'|\mathbf{s}\cdot\mathbf{r}|n,l,j,m\rangle \neq 0?$$

a. $H|n\ell jm\rangle = (-\frac{\alpha^2}{2n^2}mc^2)|n\ell jm\rangle$ $\quad L^2|n\ell jm\rangle = \ell(\ell+1)|n\ell jm\rangle$

$J^2|n\ell jm\rangle = j(j+1)|n\ell jm\rangle$ $\quad J_z|n\ell jm\rangle = m|n\ell jm\rangle$

b. $n \in \{1,2,3,\ldots\}$ $\ell \in \{0,1,2,\ldots,n-1\}$ $j \in \{\ell-\frac{1}{2}, \ell+\frac{1}{2}\}$

$m \in \{-j,\ldots,0,\ldots,j\}$

c. See above, (iii) $[L_z, p_z] = [xp_y - yp_x, p_z] = 0$

$\Rightarrow \langle 2,1,\frac{3}{2},\frac{3}{2}|L_z p_z - p_z L_z|2,1,\frac{3}{2},\frac{1}{2}\rangle = 0$

$\Rightarrow (\frac{3}{2}-\frac{1}{2})\langle 2,1,\frac{3}{2},\frac{3}{2}|p_z|2,1,\frac{3}{2},\frac{1}{2}\rangle = 0$

$\Rightarrow \langle 2,1,\frac{3}{2},\frac{3}{2}|p_z|2,1,\frac{3}{2},\frac{1}{2}\rangle = 0$

d. If $i\neq j$, then $p_i p_j$ is a cartesian componet of a rank two tensor, which is always some linear combination of rank two spherical tensors, so by the Wigner-Eckart theorem it is zero because $|j-2| \le j' \le j+2$ is not satisfied by $j = j' = \frac{1}{2}$.

If $i=j$ $\langle p_i^2\rangle = \langle p_x^2\rangle = \langle p_y^2\rangle = \langle p_z^2\rangle$ by symmetry

$= \frac{1}{3}(\langle p_x^2\rangle + \langle p_y^2\rangle + \langle p_z^2\rangle) = \frac{1}{3}\langle p^2\rangle = \frac{1}{3}2m\langle\frac{p^2}{2m}\rangle = \frac{2m}{3}\langle T\rangle$

$= -\frac{2m}{3}\langle E\rangle$ by the virial theorem

Therefore $\langle 1,0,\frac{1}{2},\frac{1}{2}|p_i p_j|1,0,\frac{1}{2},\frac{1}{2}\rangle = -\frac{2m}{3}E_1\delta_{ij}$ where $E_1 = -\frac{mk^2e^4}{2\hbar^2}$

e. $\vec{s}\cdot\vec{r}$ is a scalar, so it is the 0^{th} spherical component of a rank 0 tensor

$\Rightarrow \ell' = \ell$ and $j' = j$ and $m' = m$, but n' and n can be anything, according to the Wigner-Eckart theorem selection rules.