8. *Electricity and Magnetism* (Fall 2004)

Consider a sphere of radius a with uniform magnetization $\mathbf{M}$, pointing in the z-direction. What are the magnetic induction $\mathbf{B}$ and magnetic field $\mathbf{H}$ inside the sphere?

See Jackson Page 198.

$\vec{\nabla}\times\vec{H} = \rho_f = 0 \Rightarrow \vec{H}$ is curl free $\Rightarrow \vec{H} = -\vec{\nabla}\Phi_M$ for some scalar field Φ_M

$\vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{\nabla}\Phi_M = -\nabla^2\Phi_M$ and $\vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot(\frac{1}{\mu_0}\vec{B}-\vec{M}) = -\vec{\nabla}\cdot\vec{M} \Rightarrow \nabla^2\Phi_M = \vec{\nabla}\cdot\vec{M}$

We know the solution to poisson's equation $\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$ is $\Phi = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho}{|\vec{x}-\vec{x}'|}d^3x'$ where V is any volume that encloses all $\vec{x}$ such that $\rho(\vec{x}) \neq 0$.

Therefore $\Phi_M = -\frac{1}{4\pi}\int_V \frac{\vec{\nabla}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x'$ and lets take V as all space.

Let V_0 be the interior of the sphere and let V_0' be the complement of V_0 (V_0' is a closed set containg the boundary of the sphere).

$$\Phi_M = -\frac{1}{4\pi}\int_{V_0}\cancel{\frac{\vec{\nabla}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}}d^3x' - \frac{1}{4\pi}\int_{V_0'}\frac{\vec{\nabla}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x'$$

The first term is zero because $\vec{M}$ is constant in the interior.

$$\Phi_M = -\frac{1}{4\pi}\int_{V_0'}\left[\vec{\nabla}'\cdot\left(\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right) - \vec{M}(\vec{x}')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right]d^3x'$$

using the product rule $\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot\vec{\nabla}f$. The second term has only an infinitesimal contribution to the result because $\vec{M}$ is finite and is not nonzero over any finite subspace of V_0'.

$$\Phi_M = -\frac{1}{4\pi}\int_{V_0'}\vec{\nabla}'\cdot\left(\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)d^3x' = -\frac{1}{4\pi}\int_S \frac{\vec{M}(\vec{x}')\cdot(-\hat{n}')}{|\vec{x}-\vec{x}'|}da'$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{M\cos(\theta)}{|\vec{x}-\vec{x}'|}a^2\sin(\theta)d\theta d\phi$$

$$= \frac{Ma^2}{4\pi}\int\frac{\cos(\theta)}{|\vec{x}-\vec{x}'|}d\Omega'$$

Now we use the expansion $\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y_{\ell m}^*(\theta',\phi')Y_{\ell m}(\theta,\phi)$

$$\Phi_M = \frac{Ma^2}{4\pi}\int 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y_{\ell m}^*(\theta',\phi')Y_{\ell m}(\theta,\phi)\cos(\theta')d\Omega'$$

$$= Ma^2\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y_{\ell m}(\theta,\phi)\int Y_{\ell m}^*(\theta',\phi')\left(\sqrt{\tfrac{4\pi}{3}}Y_{10}(\theta',\phi')\right)d\Omega'$$

since $Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos(\theta)$. Now $\int Y_{\ell' m'}^*(\theta,\phi)Y_{\ell m}(\theta,\phi)d\Omega = \delta_{\ell\ell'}\delta_{mm'}$

$$\Phi_M = Ma^2\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y_{\ell m}(\theta,\phi)\sqrt{\tfrac{4\pi}{3}}\delta_{\ell 1}\delta_{m0}$$

$$= Ma^2\left(\frac{1}{3}\frac{r_<}{r_>^2}\right)Y_{10}(\theta,\phi)\sqrt{\tfrac{4\pi}{3}} = \frac{1}{3}Ma^2\frac{r_<}{r_>^2}\cos(\theta)$$

And $r_>, r_<$ are the greater and lesser between r and a, so inside $r<a$:

$$\Phi_m = \frac{1}{3}Ma^2\frac{r}{a^2}\cos(\theta) = \frac{1}{3}Mr\cos(\theta) = \frac{1}{3}Mz$$

Therefore $\vec{H} = -\vec{\nabla}\Phi_M = -\frac{1}{3}M\hat{z} = -\frac{1}{3}\vec{M}$

and $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} \Rightarrow \vec{B} = \mu_0(\vec{H}+\vec{M}) = \frac{2}{3}\mu_0\vec{M}$