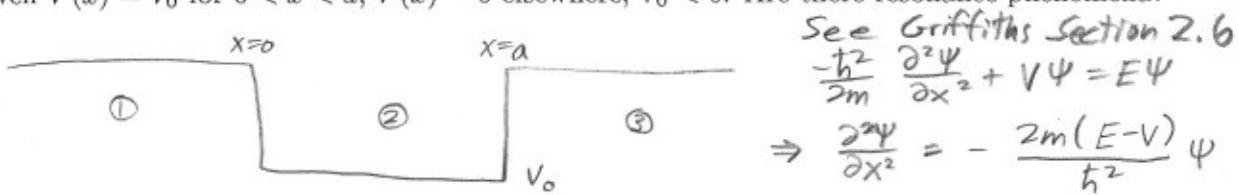


5. Quantum Mechanics (Spring 2005)

Calculate the transmission coefficient for a particle of energy $E > 0$ scattering off the 1D potential well $V(x) = V_0$ for $0 < x < a$, $V(x) = 0$ elsewhere, $V_0 < 0$. Are there resonance phenomena?



Let $K = \sqrt{\frac{2mE}{\hbar^2}}$ and $\ell = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$ which is real because $V_0 < 0 < E$

$\Rightarrow \Psi_1(x) = Ae^{ikx} + Be^{-ikx}$, $\Psi_2(x) = Ce^{ilx} + De^{-ila}$, $\Psi_3(x) = Fe^{ikx} + Ge^{-ikx}$
 $G=0$ since there is no wave coming from the right.

Since $V(x) < \infty \forall x$, we impose continuity on $\Psi(x)$ and $\Psi'(x)$:

$$\Psi_1(0) = \Psi_2(0) \Rightarrow A+B = C+D$$

$$\Psi'_1(0) = \Psi'_2(0) \Rightarrow K(A-B) = l(C-D)$$

$$\Psi_2(a) = \Psi_3(a) \Rightarrow Ce^{ila} + De^{-ila} = Fe^{ika}$$

$$\Psi'_2(a) = \Psi'_3(a) \Rightarrow l(Ce^{ila} - De^{-ila}) = KFe^{ika}$$

Now use the second two equations to solve for C and D:

$$2Ce^{ila} = (1+\frac{K}{l})Fe^{ika} \Rightarrow C = \frac{1}{2}(1+\frac{K}{l})Fe^{ika}e^{-ila}$$

$$2De^{-ila} = (1-\frac{K}{l})Fe^{ika} \Rightarrow D = \frac{1}{2}(1-\frac{K}{l})Fe^{ika}e^{ila}$$

Next use the first two equations to eliminate B and insert C, D:

$$2A = C+D + \frac{l}{K}(C-D) \Rightarrow A = \frac{1}{2}(1+\frac{l}{K})C + \frac{1}{2}(1-\frac{l}{K})D$$

$$\Rightarrow A = \frac{1}{4}(2 + \frac{K}{l} + \frac{l}{K})Fe^{ika}e^{-ila} + \frac{1}{4}(2 - \frac{K}{l} - \frac{l}{K})Fe^{ika}e^{ila}$$

$$= Fe^{ika} \cos(la) - \frac{i}{2} \frac{K^2 + l^2}{KL} Fe^{ika} \sin(la)$$

$$\Rightarrow F = Ae^{-ika} \left[\cos(la) - \frac{i}{2} \frac{K^2 + l^2}{KL} \sin(la) \right]^{-1}$$

$$\text{Therefore } T \equiv \frac{|F|^2}{|A|^2} = \left[\cos^2(la) + \frac{1}{4} \left(\frac{K^2 + l^2}{KL} \right)^2 \sin^2(la) \right]^{-1}$$

$$= \left[1 + \left(\frac{1}{4} \frac{K^4 + 2K^2l^2 + l^4}{K^2l^2} - \frac{4K^2l^2}{4K^2l^2} \right) \sin^2(la) \right]^{-1} \quad (\cos^2(la) = 1 - \sin^2(la))$$

$$= \left[1 + \frac{1}{4} \left(\frac{K^2 - l^2}{KL} \right)^2 \sin^2(la) \right]^{-1}$$

$$= \left[1 + \frac{1}{4} \left(\frac{V_0}{\sqrt{E(E-V_0)}} \right)^2 \sin^2 \left(\frac{a}{\hbar} \sqrt{2m(E-V_0)} \right) \right]^{-1}$$

$$= \left[1 + \frac{1}{4} \frac{V_0^2}{E(E-V_0)} \sin^2 \left(\frac{a}{\hbar} \sqrt{2m(E-V_0)} \right) \right]^{-1}$$

Resonance phenomena can occur if the energy is just right.

The Ramsauer-Townsend effect gives perfect transmission

$$\text{so } T=1 \Leftrightarrow \frac{a}{\hbar} \sqrt{2m(E-V_0)} = n\pi \Leftrightarrow 2m(E-V_0) = \left(\frac{n\pi\hbar}{a} \right)^2 \Leftrightarrow E = \frac{n^2\pi^2\hbar^2}{2ma^2} + V_0$$