3. *Quantum Mechanics* (Spring 2005)

A beam of particles scatters off an impenetrable sphere of radius a. That is, the potential is zero outside the sphere, and infinite inside. The wave function must therefore vanish at $r = a$.

(a) What is the S-wave ($l = 0$) phase shift as a function of the incident energy or momentum?

(b) What is the total cross section in the limit of zero incident kinetic energy?

See Sakurai pages 406-408

a. Recall $A_\ell(r) = e^{i\delta_\ell}\left[\cos(\delta_\ell) j_\ell(kr) - \sin(\delta_\ell) n_\ell(kr)\right]$

which is Sakurai (7.6.33)

The wavefunction must vanish at $r=a \Rightarrow A_\ell(r)|_{r=a} = 0$

$$\Rightarrow \cos(\delta_\ell) j_\ell(KR) - \sin(\delta_\ell) n_\ell(KR) = 0$$

$$\Rightarrow \tan(\delta_\ell) = \frac{j_\ell(KR)}{n_\ell(KR)}$$

We are trying to find the S-wave ($\ell=0$) phase shift so

$$\Rightarrow \tan(\delta_0) = \frac{j_0(KR)}{n_0(KR)} = \frac{\sin(KR)/KR}{-\cos(KR)/KR} = -\tan(KR)$$

$$\Rightarrow \delta_0 = -KR$$

b. Recall $\sigma_{tot} = \int |f(\theta)|^2 d\Omega = \frac{4\pi}{K^2}\sum_\ell (2\ell+1)\sin^2(\delta_\ell)$

which is Sakurai (7.6.18)

In the limit of zero incident kinetic energy only the $\ell=0$ term contributes so

$$\lim_{K\to 0}\sigma_{tot} = \lim_{K\to 0}\frac{4\pi}{K^2}\sin^2(\delta_0) = \lim_{K\to 0}\frac{4\pi}{K^2}\sin^2(-KR)$$

$$= \lim_{K\to 0}\frac{4\pi}{K^2}K^2R^2 = 4\pi R^2$$