10. *Electricity and Magnetism* (Spring 2006)

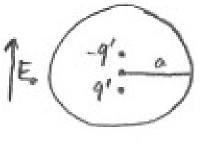
An insulated, spherical, conducting shell of radius a is in a uniform electric field E_0. If the sphere is cut into two hemispheres by a plane perpendicular to the field, find the force required to prevent the hemispheres from separating

(a) if the shell is uncharged;

This is Jackson 2.9

(b) if the total charge on the shell is Q.

$-q \bullet (z=R)$



$q \bullet (z=-R)$

Method of Images:

$z' = \frac{a^2}{R} \quad q' = -\frac{a}{R} q$

a. The strategy is:

1) Imagine the E_0 field is created by two point charges a large distance R away and use the method of images to find the potential outside the sphere

2) Find σ using $\sigma = -\epsilon_0 \frac{\partial V}{\partial n}\Big|_{r=a}$

3) Find the force using $F_z = 2\int_{hemi} E_z dq$

$= 2\int_{hemi} \left(\frac{\sigma}{2\epsilon_0}\cos(\theta)\right)\sigma dA$

$= 2\int_0^{2\pi}\int_0^{\pi/2} \frac{\sigma^2}{2\epsilon_0}\cos(\theta)\, a^2 \sin(\theta) d\theta d\phi$

where the factor of 2 comes from having to push both hemispheres in order to hold them together.

1) $E_0 = 2\frac{1}{4\pi\epsilon_0}\frac{q}{R^2} \Rightarrow q = 2\pi\epsilon_0 R^2 E_0$

Recall $\vec{p} \equiv \int \vec{r}' \rho(\vec{r}') d^3r' \Rightarrow \vec{p}$ points toward positive charge, unlike fields

$\vec{p} = |2z'|\cdot|q'|\hat{z} = 2\frac{a^3}{R^2} q\hat{z} = 4\pi\epsilon_0 a^3 E_0 \hat{z}$

Remember that $V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$ so

$V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{4\pi\epsilon_0 a^3 E_0}{r^2}\cos(\theta) = \frac{a^3}{r^2} E_0 \cos(\theta)$

$V_{field} = -\int \vec{E}\cdot d\vec{r} = -E_0 r\cos(\theta)$

$V = V_{dip} + V_{field} = -\left(r - \frac{a^3}{r^2}\right) E_0 \cos(\theta)$

2) $\sigma = -\epsilon_0 \frac{\partial V}{\partial n}\Big|_{r=a} = -\epsilon_0 \frac{\partial V}{\partial r}\Big|_{r=a} = \epsilon_0\left(1 + 2\frac{a^3}{r^3}\right) E_0 \cos(\theta)\Big|_{r=a}$

$= 3\epsilon_0 E_0 \cos(\theta)$ which is Jackson (2.15)

3) $F_z = 9\epsilon_0 E_0^2 a^2 (2\pi)\int_0^{\pi/2}\cos^3(\theta)\sin(\theta)\, d\theta$ Let $x = \cos(\theta)$

$= 9\epsilon_0 E_0^2 a^2 (2\pi)\int_0^1 x^3 dx = \frac{9}{2}\pi\epsilon_0 E_0^2 a^2$

b. In part a, the surface was an equipotential, so an additional charge Q will spread out uniformly: $\sigma \to 3\epsilon_0 E_0 \cos(\theta) + Q/4\pi a^2$

$F_z = \frac{2\pi a^2}{\epsilon_0}\int_0^{\pi/2}\left[9\epsilon_0 E_0 \cos^2(\theta) + 2\frac{3\epsilon_0 E_0 Q}{4\pi a^2}\cos(\theta) + \frac{Q^2}{16\pi^2 a^4}\right]\cos(\theta)\sin(\theta) d\theta$

$= \frac{9}{2}\pi\epsilon_0 E_0^2 a^2 + 3E_0 Q\int_0^{\pi/2}\cos^2(\theta)\sin(\theta) d\theta + \frac{Q^2}{8\pi a^2 \epsilon_0}\int_0^{\pi/2}\cos(\theta)\sin(\theta) d\theta$

$= \frac{9}{2}\pi\epsilon_0 E_0^2 a^2 + E_0 Q + \frac{Q^2}{16\pi a^2 \epsilon_0}$

But the middle term is the Q charge interacting with the field, which is in the same direction for both hemispheres, so

$F = \frac{9}{2}\pi\epsilon_0 E_0^2 a^2 + \frac{Q^2}{16\pi a^2 \epsilon_0}$