Green's Functions

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1 Differential Equations

Suppose we want to solve a differential equation

$$L_{\mathbf{x}}\phi(\mathbf{x}) = \lambda(\mathbf{x}),$$

where $L_{\mathbf{x}}$ is a linear operator, $\lambda(\mathbf{x})$ is an inhomogeneous term, and $\phi(\mathbf{x})$ is the function that we would like to solve for. In general, $\lambda(\mathbf{x})$ can be quite complicated, which could make it difficult to find a solution. However, using Green's functions it is possible to take advantage of the linearity of $L_{\mathbf{x}}$ to replace this problem with an integral and a simpler differential equation.

Theorem 1.1. *If we define a Green's function $G(\mathbf{x}, \mathbf{x}')$ so that*

$$L_{\mathbf{x}}G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'),$$

then the solution to $L_{\mathbf{x}}\phi(\mathbf{x}) = \lambda(\mathbf{x})$ is

$$\phi(\mathbf{x}) = \int G(\mathbf{x}, \mathbf{x}')\lambda(\mathbf{x}')\ d^3x'.$$

Proof. We can verify this by plugging the claimed solution into the differential equation.

$$L_{\mathbf{x}}\phi(\mathbf{x}) = L_{\mathbf{x}} \int G(\mathbf{x}, \mathbf{x}')\lambda(\mathbf{x}')\ d^3x'$$

$$L_{\mathbf{x}}\phi(\mathbf{x}) = \int L_{\mathbf{x}}G(\mathbf{x}, \mathbf{x}')\lambda(\mathbf{x}')\ d^3x'$$

$$L_{\mathbf{x}}\phi(\mathbf{x}) = \int \delta(\mathbf{x} - \mathbf{x}')\lambda(\mathbf{x}')\ d^3x'$$

$$L_{\mathbf{x}}\phi(\mathbf{x}) = \lambda(\mathbf{x})$$

so $\phi(\mathbf{x})$ satisfies the differential equation. □

So it is not necessary to solve the original differential equation. We can instead solve the differential equation for $G(\mathbf{x}, \mathbf{x}')$, which has a simple delta function inhomogeneous term, and then do an integral. Doing the integral does require an extra step, but it is often worth the cost in order to simplify the differential equation. But there is an even more important benefit of Green's functions that we will see in the next section.

2 Poisson's Equation

One of the most efficient ways of calculating electric fields in electrostatics is to solve Poisson's equation for the potential and then take the negative gradient to get the field. Let's try using Green's functions to solve Poisson's equation

$$\nabla^2\phi(\mathbf{x}) = -\frac{\rho(\mathbf{x})}{\epsilon_0}.$$

Theorem 1.1 says that we should define $G(\mathbf{x}, \mathbf{x}')$ so that

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}').$$

Mathematically, this is not too easy to solve. Fortunately we can use our knowledge of electrostatics to see the solution. This is still a form of Poisson's equation, so $G(\mathbf{x}, \mathbf{x}')$ must be the potential of a charge distribution that looks like a delta function. Well, that's just a point charge. So if we treat $\mathbf{x}'$ as a constant parameter,

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = -\frac{\rho_G(\mathbf{x}, \mathbf{x}')}{\epsilon_0} = \delta(\mathbf{x} - \mathbf{x}')$$
$$\Rightarrow \rho_G(\mathbf{x}, \mathbf{x}') = -\epsilon_0 \delta(\mathbf{x} - \mathbf{x}')$$

This means that G is the potential of a point charge of charge $q = -\epsilon_0$ at $\mathbf{x}'$, so

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi\epsilon_0} \frac{q}{|\mathbf{x} - \mathbf{x}'|} = -\frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{x}'|}$$

Now Theorem 1.1 tells us the solution for $\phi(\mathbf{x})$

$$\phi(\mathbf{x}) = \int G(\mathbf{x}, \mathbf{x}') \lambda(\mathbf{x}') \, d^3x' = \int \left(-\frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \left(-\frac{\rho(\mathbf{x}')}{\epsilon_0} \right) \, d^3x' = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d^3x'.$$

But if that was all that Green's functions did for us, then they wouldn't be too impressive because we've all seen that same derivation before, just without any mention of Green's functions. So why are we going through all this trouble?

The reason is boundary conditions. The crucial observation is the following. *No matter what function you pick for $G(\mathbf{x}, \mathbf{x}')$, as long as it satisfies the defining differential equation, then the solution for $\phi(\mathbf{x})$ given in Theorem 1.1 works.* This means that we can always add a homogeneous solution

$$L_{\mathbf{x}} G_0(\mathbf{x}, \mathbf{x}') = 0$$

to the particular solution in $G(\mathbf{x}, \mathbf{x}')$. Then it is fine to use the new form $G'(\mathbf{x}, \mathbf{x}') = G(\mathbf{x}, \mathbf{x}') + G_0(\mathbf{x}, \mathbf{x}')$ in the solution of $\phi(\mathbf{x})$ and if you choose the homogeneous solution appropriately, you can make the integral for $\phi(\mathbf{x})$ simplify drastically. But it's not clear how we can do this until we use Green's theorem to rewrite the integral solution of Poisson's equation.

3 Green's Second Identity

Our objective is to find a solution of Poisson's equation that only requires integration over a finite region of interest rather than over all space. This is because many problems in electrostatics are specified with conditions on finite boundaries, for example the potential on the inside surfaces of a container. Green's second identity is a simple consequence of the divergence theorem and some vector identities. It will allow us to write a solution of Poisson's equation in terms of an integral over a finite volume plus an integral over the bounding surface. The Green's function can then be chosen so as to cause one of the terms in the surface integral to drop out.

Theorem 3.1. *(Green's Second Identity) Consider a closed volume V bounded by a surface S. Let ψ_1 and ψ_2 be scalar fields defined on V and S. Then*

$$\int_V \left(\psi_1 \nabla^2 \psi_2 - \psi_2 \nabla^2 \psi_1 \right) \, d^3x = \oint_S \left(\psi_1 \frac{\partial \psi_2}{\partial n} - \psi_2 \frac{\partial \psi_1}{\partial n} \right) \, da$$

Proof. The divergence theorem says

$$\int_V \nabla \cdot \mathbf{A} \, d^3x = \oint_S \mathbf{A} \cdot \mathbf{n} \, da.$$

Now we pull a trick and let $\mathbf{A} = \psi_1 \nabla \psi_2$.

$$\int_V \nabla \cdot (\psi_1 \nabla \psi_2) \, d^3x = \oint_S (\psi_1 \nabla \psi_2) \cdot \mathbf{n} \, da.$$

Next we use the product rule and the definition of the normal derivative to obtain Green's first identity,

$$\int_V \left(\psi_1 \nabla^2 \psi_2 + \nabla\psi_1 \cdot \nabla\psi_2\right) d^3x = \oint_S \psi_1 \frac{\partial \psi_2}{\partial n}\, da.$$

Finally, if we interchange ψ_1 and ψ_2 and subtract the resulting equation from the previous equation, then the $\nabla\psi_1 \cdot \nabla\psi_2$ term will cancel out, yielding Green's second identity,

$$\int_V \left(\psi_1 \nabla^2 \psi_2 - \psi_2 \nabla^2 \psi_1\right) d^3x = \oint_S \left(\psi_1 \frac{\partial \psi_2}{\partial n} - \psi_2 \frac{\partial \psi_1}{\partial n}\right) da$$

□

So let's use this theorem to find a new way to write the solution of Poisson's equation. We just have to choose ψ_1 and ψ_2 appropriately. Let $\psi_1 = \phi(\mathbf{x})$ and $\psi_2 = G(\mathbf{x}, \mathbf{x}')$ where $\mathbf{x}'$ is taken to be a constant parameter,

$$\int_V \left(\phi(\mathbf{x})\nabla^2 G(\mathbf{x}, \mathbf{x}') - G(\mathbf{x}, \mathbf{x}')\nabla^2 \phi(\mathbf{x})\right) d^3x = \oint_S \left(\phi(\mathbf{x})\frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n} - G(\mathbf{x}, \mathbf{x}')\frac{\partial \phi(\mathbf{x})}{\partial n}\right) da$$

$$\int_V \left(\phi(\mathbf{x})\delta(\mathbf{x}, \mathbf{x}') - G(\mathbf{x}, \mathbf{x}')\lambda(\mathbf{x})\right) d^3x = \oint_S \left(\phi(\mathbf{x})\frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n} - G(\mathbf{x}, \mathbf{x}')\frac{\partial \phi(\mathbf{x})}{\partial n}\right) da$$

So if $\mathbf{x}'$ is in the volume V, then

$$\phi(\mathbf{x}') = \int_V G(\mathbf{x}, \mathbf{x}')\lambda(\mathbf{x})\, d^3x + \oint_S \left(\phi(\mathbf{x})\frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n} - G(\mathbf{x}, \mathbf{x}')\frac{\partial \phi(\mathbf{x})}{\partial n}\right) da$$

Finally we can see the real reason for using Green's functions. Suppose we have Dirichlet boundary conditions on some surface. Then we can let that surface be S (so V will be the enclosed volume) and then choose a homogeneous solution to add to $G(\mathbf{x}, \mathbf{x}')$ so that it will be zero on the whole surface S. Then the last term in the surface integral is always zero, and thus there is no need to know what the normal derivative of the potential is. You could derive this same solution without any reference to Green's functions, but then the fact that you can add an inhomogeneous solution to all the places where G shows up would not be obvious at all. Green's functions make it clear that this can be done, and that makes it much easier to solve boundary value problems.

4 Example

Suppose we have a potential $V(x, y)$ specified on an infinite plane $z = 0$ and we need to calculate the potential everywhere in the half-space given by $z > 0$. Since the potential is specified, we have Dirichlet boundary conditions for Poisson's equation. The general solution of the corresponding Green's function differential equation, $\nabla^2 G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$ is

$$G(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi}\frac{1}{|\mathbf{x} - \mathbf{x}'|} = -\frac{1}{4\pi}\frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} + G_0(\mathbf{x}, \mathbf{x}')$$

where $\nabla^2 G_0(\mathbf{x}, \mathbf{x}') = 0$ in the region of interest, $z > 0$. By looking at the result of the last section, we notice that we know $\phi(\mathbf{x})$, but not $\frac{\partial \phi(\mathbf{x})}{\partial n}$, so we would like to use the freedom of $G_0(\mathbf{x}, \mathbf{x}')$ to take the coefficient of $\frac{\partial \phi(\mathbf{x})}{\partial n}$ to zero which will allow us to get away with our lack of knowledge. The coefficient is $G(\mathbf{x}, \mathbf{x}')$ and we need it to be zero on the surface $z = 0$. This can be accomplished by setting $G_0(\mathbf{x}, \mathbf{x}')$ to be the potential of a point charge at the location given by reflecting $\mathbf{x}'$ about the $z = 0$ plane. This won't affect the answer because $\nabla^2 G_0(\mathbf{x}, \mathbf{x}')$ is only non-zero at the location of the image charge which is outside of the region of interest. Therefore,

$$G(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi}\left(\frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} + \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z + z')^2}}\right)$$

The equation to solve is now

$$\phi(\mathbf{x}') = \int_V G(\mathbf{x}, \mathbf{x}') \left(-\frac{\cancel{\rho(\mathbf{x})}}{\epsilon_0} \right) \, d^3x + \oint_S \left(\phi(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n} - \cancel{G(\mathbf{x}, \mathbf{x}')} \frac{\partial \phi(\mathbf{x})}{\partial n} \right) \, da$$

$$\phi(\mathbf{x}') = \oint_S \phi(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n} \, da$$

From this point it is just a matter of computing derivatives and the integral.